§3 Group Representations

Let G be a finite gp, $k=\bar{k}$ field, char $k = p \geq 0$

Def^n
A representation of G is a hom^m $\rho : G \to GL(V)$ for some k-space V.

$\rightsquigarrow$ G-action on V

We say V is faithful if $gv = v \ \forall v \in V \Rightarrow g = 1$.

Simple $\longleftrightarrow$ irreducible. $Irr_k(G) :=$ Set of iso. classes of irred. kG-modules.

Def^n
$V^G := \{ v \in V \mid gv = v \ \forall g \in G \}$. If $V^G = V$ say V is trivial, "the trivial module" is the 1-dim space k with $k^G = k$.

Ex
kG is a kG-module, often called the regular representation of G over k.

Ex
$\sigma := \sum_{g \in G} g \in kG$, then $k\sigma \leq kG$, $k\sigma \cong k$.

$g(xH) = (gx)H$

Ex
Let $H \leq G$, the permutation module of G on H is $k(G/H)$ with obvious G-action

Ex $G = S_n$, $H = Stab_G(1) \cong S_{n-1} \leq G$. Then $G \circlearrowright G/H$ is iso. to natural action on $\Omega = \{1, \ldots, n\}$

Typically regard this as

$$V = Span_k\{v_1, \ldots, v_n\} \text{ where } g v_i = v_{ig}$$

$V^G = k\sum_{i=1}^{n} v_i$. If $p = 0$ or $p > n$, $W = V/V^G$ is irred.

$V = k\sum_{i=1} V_i$, if $p=0$ or $p>$ ...

Th^m (Maschke's Th^m)

kG is semisimple $\iff p \nmid |G|$.

Proof Suppose $p \mid |G|$. Then $p>0$ is prime. We know $k\sigma \cong k \leq kG$, if kG is semisimple then $kG \cong k \oplus U$. By Cor 2.27, U has no submodules iso. to k, in any comp. series of kG, k appears only once.

Take hom^m $G \twoheadrightarrow 1$, extend to $kG \twoheadrightarrow k$ with kernel $\Delta(G) = \left\{ \sum_{g\in G} a_g g \;\middle|\; \sum_{g\in G} a_g = 0 \right\}$

Now, $\frac{kG}{\Delta(G)} \cong k$ (as $g \cdot 1 = 1 + (g-1) \in 1 + \Delta(G)$)

As $p \mid |G|$, $k\sigma \leq \Delta(G)$, so

$$0 \leq k\sigma \leq \Delta(G) \leq kG$$

$$k \qquad \Delta(G)/k \qquad k. \quad \Rightarrow kG \text{ not semisimple.}$$

Now suppose $p \nmid |G|$.
So $|G|$ is invertible in k.
Let V be a kG-module, $U \leq V$. Suff. to show $\exists\, W \leq V$ s.t. $V = U \oplus W$.
Let $\pi : V \twoheadrightarrow U$ be a proj^n map. So as vector spaces, $V = U \oplus \ker\pi$.

Let $\pi' := \frac{1}{|G|} \sum_{g\in G} \pi^g$

Let $h \in G$, $v \in V$, then
$$\pi'(hv) = \frac{1}{|G|} \sum_{g\in G} \pi^g hv = \frac{1}{|G|} \sum_{g\in G} hh^{-1}g^{-1}\pi ghv = h\frac{1}{|G|}\sum_{x\in G} \pi^x v = h\pi'(v). \quad \square$$

Def^n

$g \in G$ is p-regular if $p \nmid |g|$, g^G p-reg $\iff p \nmid |g|$.

$\underline{\text{Th}^{\underline{m}}}$

$|\mathrm{Irr}_k G|$ is the number of p-regular classes in G.

Cor if $p \nmid |G|$, $|\mathrm{Irr}_k G| = k(G) = \mathrm{N}^{\underline{o}}$ classes in G.

Cor

If G is a p-gp then $\mathrm{Irr}_k(G) = \{k\}$.

Ex $G = \langle g \rangle$, $|g| = n = p^a r$ for $p \nmid r$.

Then $x^r - 1$ is separable over k with r roots. Fix one, λ. Since $\lambda^n = (\lambda^r)^{p^a} = 1$ this is a form a 1-dim space where g^i acts as mult. by λ^i.

kG-module. iso. type determined by λ.

Since G has r p-rg. classes, this is all of $\mathrm{Irr}_k G$.

Note: $x^n - 1 = (x^r - 1)^{p^a}$ So all n^{th} roots in k are r^{th} roots.

Ex $G = SL_2(p)$. Then G has p classes of p-rg. elts., so $|\mathrm{Irr}_k G| = p$.

Let V be the natural kG-module. $V = \mathrm{Span}_k\{X, Y\}$ $X = \begin{pmatrix}1\\0\end{pmatrix}$, $Y = \begin{pmatrix}0\\1\end{pmatrix}$.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have $gX = aX + cY$, $gY = bX + dY$

This defines an action of G on $k[X,Y]$.

Let V_n denote the subspace of $k[X,Y]$ of homogeneous polynomials of degree $n-1$.

$$V_n = \mathrm{Span}_k\{X^{n-1}, X^{n-2}Y, \dots, XY^{n-2}, Y^{n-1}\}$$

So $V_1 \cong k$, $V_2 \cong V$, $\dim V_n = n$.

Claim: $\mathrm{Irr}_k(G) = \{V_1, \dots, V_p\}$.

Let $1 \leq n < p$, $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

for $0 \leq i \leq n$ let W_{i+1} be the $(i+1)$-dim. subspace of V_{n+1} spanned by

For $0 \leqslant i \leqslant n$, let W_{i+1} be the $(i+1)$-dim. Subspace of V_{n+1} Spanned by

$$\{X^i Y^{n-i}, X^{i-1} Y^{n-i+1}, \ldots, X Y^{n-1}, Y^n\}, \qquad W_0 := 0.$$

So we get

$$0 = W_0 \lneq W_1 \lneq \cdots \lneq W_n \leqslant W_{n+1} = V_{n+1}$$

We show: for $0 < i \leqslant n$:
i) W_i is a $K\langle g\rangle$-Submodule of V_{n+1}
ii) $W_i / W_{i-1} \cong K$ as a $K\langle g\rangle$-module.
iii) each element of $W_i \setminus W_{i-1}$ generates W_i as a $K\langle g\rangle$-module.

Proof
for $i=1$, note $gX = X+Y$, $gY = Y$. So $gY^n = Y^n$ So g acts trivially on W_1.
Suppose claim holds for i.

$$gX^iY^{n-i} = (X+Y)^i Y^{n-i} \underset{\text{binomial}}{=} X^iY^{n-i} + \binom{i}{1} X^{i-1} Y^{n-i+1} + u$$

where $u \in W_{i-1}$. Since $gX^iY^{n-i} \in W_{i+1}$ and any such elt. may be written as $aX^iY^{n-i} + x$ for $x \in W_i$, and W_i is a $K\langle g\rangle$-Submodule of V_{n+1}.

Since $\dim(W_{i+1}/W_i) = 1$ and $gX^iY^{n-i} - X^iY^{n-i} \in W_i$ we have $W_{i+1}/W_i \cong K$ as $K\langle g\rangle$-modules

Take $v \in W_{i+1} \setminus W_i$, then $v = aX^iY^{n-i} + x$ for $x \in W_i$, $a \neq 0$ So

$$(g-1)v = a(g-1)X^iY^{n-i} + (g-1)x$$

where $(g-1)x \in W_{i-1}$ and so $(g-1)v \in W_i \setminus W_{i-1}$. By induction, v generates W_{i+1}. $\square$

In particular, X^n generates V_{n+1} as a $K\langle g\rangle$-module. As $|g| = p$, the only irred. $K\langle g\rangle$-module is K.
Thus $\mathrm{Soc}\, V_{n+1} = V_{n+1}^{\langle g\rangle} = \mathrm{Span}\{Y^n\}$. Similarly, Y^n generates V_{n+1} as a $K\langle h\rangle$-module.
Now, let $0 \neq W \leqslant V_{n+1}$. Then W is a $K\langle g\rangle$-Submodule, thus $\mathrm{Soc}\, V_{n+1} \leqslant W$. But then W contains $K\langle h\rangle \cdot \mathrm{Span}\{Y^n\} = V_{n+1}$. So $V_{n+1} \in \mathrm{Irr}_K(E)$.

Now, let $0 \neq W \leq V_{n+1}$ ...

W containing $K\langle h\rangle \cdot \mathrm{Span}\{Y^n\} = V_{n+1}$. So $V_{n+1} \in \mathrm{Irr}_K(E)$.

Defn

Let $H \leq G$, we denote the restriction of V to H by V_H.

Thm (Clifford)

If V is a semisimple KG-module, $N \trianglelefteq G$, then V_N is semisimple.

Proof Suff. to let V be irred., let $W \leq V_N$. Then for $g \in G$, $gW \leq V_N$. Since

$$n(gW) = gg^{-1}ngW = gn^gW = gW. \quad \forall n \in N.$$

Let $U \leq V_N$ be irred. Then $\sum_{g\in G} gU$ is a semisimple submodule of V_N, but it is also a KG-module.

But then it must be all of V_N. $\square$

Cor (to Maschke's Thm)

Suppose $p \nmid |G|$. Then every KG-module is projective.

Thm

There is a 1-1 correspondence between projective, indecomposable KG-modules (PIMs) and iso. classes of irred. KG-modules, given by $P \leftrightarrow hd(P) = P/\mathrm{rad}P$.

P, Q PIMs, $hdP \cong hdQ \Rightarrow P \cong Q$.

Defn Let $V \in \mathrm{Irr}_K G$. The unique PIM P s.t. $V \cong hdP$ is called the projective cover of V, $\mathcal{P}(V)$.

Cor

$$KG \cong \bigoplus_{V \in \mathrm{Irr}_K G} \mathcal{P}(V)^{\oplus \dim V}$$

Lemma

Let V be a KG-module with $hd V \cong hd P$ for some PIM P. Then $\exists \varphi : P \twoheadrightarrow V$.

Lemma
Let V be a kG-module with $\mathrm{hd}\, V \cong \mathrm{hd}\, P$ for some PIM P. Then $\exists \psi : P \twoheadrightarrow V$.

Th^{m}
If P is proj$^{'ve}$ then so is P_H for any $H \leq G$. "$\square$"

"Proof": follows from $(kG)_H \cong (kH)^{\oplus [G:H]}$.

Cor
If $Q \in \mathrm{Syl}_p G$ with $|Q| = p^a$, then every projective kG-module has dimension divisible by p^a.

Proof kQ is indecomposable, every proj$^{'ve}$ kQ-module is free, and so has dim divisible by

$$\dim kQ = |Q| = p^a.$$

$\square$

Ex $G = \langle g \rangle$. $|g| = n = p^a r$ $\quad p \nmid r$. Let V be a kG-module afforded by $\varphi : G \to GL(V)$.
Then the eigenvalues of $\varphi(g)$ are n^{th} roots of unity. In fact,

$$V = V_1 \oplus \cdots \oplus V_s$$

where each V_i is a Jordan block, i.e. $\varphi(g)$ acts on V_i as
$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$
for some eigenval. λ of $\varphi(g)$

each V_i is indecomp.
As such, each indecomp. kG-module is a Jordan block for some n^{th} root of 1, λ.
Suppose $V = V_i$ is indecomp. w/ basis $\{v_1, \ldots, v_m\}$ corr. to the matrix above. As before, λ is actually an n^{th} root of 1.
If $\lambda_1, \ldots, \lambda_r$ be the r^{th} roots of unity. Then

$$\varphi(g)^r - Id_V = (\varphi(g) - \lambda_1 Id_V)(\varphi(g) - \lambda_2 Id_V) \cdots (\varphi(g) - \lambda_r Id_V)$$

and so $\varphi(g)^r - Id_V = (\varphi(g) - \lambda Id_V) S$ where S is a nonsingular lin. transformation of V commuting with $\varphi(g)$. Also,

$$(\varphi(g)^r - Id)^{p^a} = (\varphi(g) - \lambda Id_V)^{p^a} S^{p^a}$$

with $\varphi(g)$. Also,

$$0 = \varphi(g)^n - Id_V = \left(\varphi(g)^r - Id_V\right)^{p^a} = \left(\varphi(g) - \lambda Id_V\right)^{ra} \quad S^{p^a}$$

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix}$$

and so $(\varphi(g) - Id_V)^{p^a} = 0$.

But, $(\varphi(g) - \lambda Id_V) v_i = v_{i+1}$ $\forall i < m$ and $(\varphi(g) - \lambda Id_V) v_m = 0$,

So $m \leq p^a$.

$\rightsquigarrow$ n possible structures for V (r choices of λ, p^a choices of $m = \dim V$)

We can see $g^r - 1 \in KG$ is nilpotent, thus (as G is commutative), $g^r - 1 \in \operatorname{rad} KG$.

$$(g^r - 1) V = \operatorname{rad} V = \operatorname{span}_K \{v_2, \dots, v_m\}$$

Iterating: $\operatorname{rad}^i V$ shrinks by 1 dimension at a time.

So each indecomp. KG-module looks like

$$\begin{array}{c} V_\lambda \\ V_\lambda \\ | \\ V_\lambda \end{array} \qquad \text{for some } \lambda.$$

w/ comp. length at most p^a.
