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Defn

Let G be a group and $H \leq G$. Then H is a trivial-intersection subgp (TI-subgp) if for any $g \in G$, $H \cap H^g = 1$ or $H \cap H^g = H$.

Thm

Let $P \in Syl_p G$ be a TI-subgp and $L := N_G(P)$. Then there is a 1-1 corr. between isomorphism classes of non-proj've indecomp. kG-modules and (*) ... kL-modules, such that if V and U are such corr. modules, then

$$V_L \cong U \oplus Q \qquad Ind_L^G U \cong V \oplus R$$

for Q and R proj've kL- and kG-modules, resp..

Now, write $\mathrm{Ind}_L^G U = V_1 \oplus V_2 \oplus \dots \oplus V_n$ as a decomp. into indecomp. factors.

Since L contains a Sylow p-subgp of G, V_i is proj^ve iff $(V_i)_L$ is.

So all V_i bar one are proj^ve. Wlog let $V = V_1$ be the non-proj^ve one.

So we have $\mathrm{Ind}_L^G U \cong V \oplus Q$ for Q proj^ve and $V_L \cong U \oplus R$ for proj^ve R.

Now, suppose V is a non-proj^ve indecomp. KG-module. As L contains a Sylow p-subgp of G every KG-module is rel. L-proj^ve, in particular $\exists$ indecomp. KL-module U s.t. $V \mid \mathrm{Ind}_L^G U$.

Thus $V_L \cong U \oplus R$ for R proj^ve. This yields a map from the set of iso. classes of non-proj^ve indecomp. KG-modules and the same set for KL, and its inverse. $\square$.

<u>Lemma</u>

Suppose every PIM for G is uniserial. Then every indecomp. KG-module is uniserial.

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Proof

Suppose M is an indecomp. kG-module with $V \leq M$ irreducible and let M' be the submodule of M which is max. s.t. $M' \cap V = 0$. By maximality, $\mathrm{Soc}(M/M') = V$.

By the dual of Lemma 3.20, M/M' embeds in $\mathcal{P}(V)$ and is thus uniserial.

Thus head(M/M') is simple, so $P := \mathcal{P}(M/M')$ is a PIM, thus uniserial.

Proof

Let $V \in \mathrm{Irr}_k G$, $Q \in \mathrm{Syl}_p G$, $P := \mathcal{P}(V)$. Then P_Q is proj^{ve}, thus free of rank $\dim V$.

By Lemma 3.23, the radical series of P and P_Q coincide and since kQ has radical length $|Q|$ with each layer of P_Q has dimension $\dim V$, the same must be true of P.

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Let $V := k$ be the trivial module. By the above, $W := \text{rad}\,P / \text{rad}^2 P$ is 1-dim, thus irred..

If $U := P/\text{rad}^2 P$, we have a non-split extension

$$0 \to W \to U \to k \to 0$$

i.e. $U \sim \overset{k}{W}$

Thus $V \otimes U$ is uniserial with comp. factors $V \otimes W$ and V. In fact, $V \otimes U \cong P/\text{rad}^2 P$.

Now, suppose M is an indecomp. kG-module with head $M \cong V$. Since M is a quotient of P, $M \cong V$ or $\text{rad}\,M / \text{rad}^2 M \cong V \otimes W$.

So, we know P has comp. length $|Q|$, head V and $\text{rad}\,P/\text{rad}^2 P \cong V \otimes W$. Then $\text{rad}\,P_\lambda \leq P$ with $\text{hd}(\text{rad}\,P) \cong V \otimes W$ and comp. length $|Q|-1$. By above, either $\text{rad}\,P \cong V$ (and $|Q| = 2$)

With $\mathrm{hd}(\mathrm{rad}\, P) \cong V \otimes W$ and comp. length $|Q|-1$. By above, either $\mathrm{rad}\, P = V$ (…)

or $\mathrm{rad}^2 P / \mathrm{rad}^3 P \cong (V \otimes W) \otimes W$. Iterating this, $\mathrm{rad}^{i-1} P / \mathrm{rad}^i P \cong V \otimes W^{\otimes (i-1)}$. $\square$

<u>Corollary</u>

Suppose G has a cyclic normal Sylow p-subgp. Then every indecomp. KG-module is uniserial.

<u>Ex</u>

$G := SL_2(p)$.

Let $P = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c \in \mathbb{F}_p \right\} \in \mathrm{Syl}_p G$. Since $|P| = p$, clearly P is a TI-subgp. Let $L := N_G(P)$.

Then $L = \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \mid c \in \mathbb{F}_p, a \in \mathbb{F}_p^\times \right\}$. Let $g := \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \in L$.

Define $\varphi : L \to \mathbb{F}_p^\times$ by $\varphi(g) = a$. Then $\ker \varphi = P$ and $L/P \cong C_{p-1}$. So $\mathrm{Irr}_K L$ is made up of

$p-1$ 1-dim. modules.

Recall that $Soc((V_i)_P)$ has dim. 1 and basis y^{i-1}. Since P lies in the kernel of all simple kL-modules this remains true for L. We see that

$$Soc((V_i)_L) \cong U_{-(i-1)} \quad \text{So } (V_i)_L \sim \begin{matrix} U_{i-1} \\ U_{i-3} \\ \vdots \\ U_{-i+1} \end{matrix}$$

Lemma ①

Let $1 \leq i < p-1$. Then there exists a non-split extension

$$0 \to V_{p-i-1} \to V \to V_i \to 0$$

Lemma ②

Let $1 < i \leq p-1$. Then there is a non-split extension

$$0 \to V_{p+1-i} \to V \to V_i \to 0.$$

§5 Blocks

Thm

Let A be an algebra. Then A has a unique decomp. into indecomposable subalgebras.

$$A = A_1 \oplus \dots \oplus A_n$$

These indecomp. summands A_i are called the blocks of A and each is a two-sided ideal of A.

If $a_i \in A_i$, $a_j \in A_j$, then $a_i a_j \in A_i \cap A_j = \delta_{ij} A_j$ $\quad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{o/w} \end{cases}$