Modular Representation Theory Exam — TCC 2024/25

Return your solutions to me at J.P.Saunders@bristol.ac.uk by 10th January 2025.

For all questions below, assume that G is a finite group and k is an algebraically closed field of characteristic p.

1. Let A be an algebra and let M be an A-module. Show that the radical length and socle length of M coincide.

2. Let A be an algebra and let $\varphi\colon M \to N$ be an injective homomorphism of A-modules. Show that $\varphi(M)$ is a direct summand of N if and only if there exists $\psi\colon N \to M$ such that $\psi\varphi$ is the identity on M, *i.e.* $\psi(\varphi(m)) = m$ for all $m \in M$.

3. Suppose that V is an irreducible kG-module and W is a 1-dimensional kG-module. Show that $V \otimes W$ is irreducible.

4. Let V be an irreducible kG-module.
   (a) Show that the multiplicity of V as a composition factor of the kG-module U is $\dim \operatorname{Hom}_G(\mathcal{P}(V), U)$.
   (b) Suppose also that $W \in \operatorname{Irr}_k G$. Deduce that the multiplicity of V as a composition factor of $\mathcal{P}(W)$ is the same as the multiplicity of W as a composition factor of $\mathcal{P}(V)$.

5. Suppose that V is a relatively H-projective kG-module. Show that $V \otimes U$ is relatively H-projective for any kG-module U.

6. Let $P \in \operatorname{Syl}_p G$ be cyclic and normal in G and let $W := \operatorname{rad} \mathcal{P}(k)/\operatorname{rad}^2 \mathcal{P}(k)$, where k denotes the trivial kG-module. Show that $U, V \in \operatorname{Irr}_k G$ lie in the same block if and only if $V \cong U \otimes W^{\otimes n}$ for some n.

7. (a) Prove that the blocks of $\mathrm{SL}_2(p)$ for $p > 2$ are as stated in Example 5.5.
   (b) Show that the Brauer trees for these blocks are as stated in Example 6.8 for $p > 2$.

8. Prove Corollary 5.20: Let G be a finite group and B a block of kG. Then B is a simple algebra if and only if B has defect zero.

9. Let B be a block with the below Brauer tree. Determine the structure of the projective indecomposable modules corresponding to the irreducible modules U, V and W in the cases $(m, n) = (3, 1)$ and $(m, n) = (1, 3)$.