Modular Representation Theory Exam — TCC 2024/25

Return your solutions to me at J.P.Saunders@bristol.ac.uk by 10th January 2025.

For all questions below, assume that G is a finite group and k is an algebraically closed field of characteristic p. Further, any algebras A are finite-dimensional over k and any A-modules are finitely generated.

1. Let A be an algebra and let M be an A-module. Show that the radical length and socle length of M coincide.

   Solution:
   Suppose that the socle length of M is s and the radical length of M is r. Then $\operatorname{soc}^s M = M$ and $M/\operatorname{soc}^{s-1} M$ is a semisimple module. Thus $\operatorname{rad} M \leq \operatorname{soc}^{s-1} M$. Since $\operatorname{soc}(\operatorname{rad} M) = \operatorname{soc} M \cap \operatorname{rad} M$ we thus have that the socle length of $\operatorname{rad} M$ is $s-1$. Iterating this, we see that the socle length of $\operatorname{rad}^i M$ is $s-i$, and in particular $\operatorname{rad}^r M = 0$ has socle length $s-r$ so that $s = r$. ∎

2. Let A be an algebra and let $\varphi\colon M \to N$ be an injective homomorphism of A-modules. Show that $\varphi(M)$ is a direct summand of N if and only if there exists $\psi\colon N \to M$ such that $\psi\varphi$ is the identity on M, *i.e.* $\psi(\varphi(m)) = m$ for all $m \in M$.

   Solution:
   First suppose that $\varphi(M) \mid N$, so that $N \cong M \oplus M'$ for some submodule $M' \leq N$. Then the natural projection map $\pi\colon N \to \varphi(M)$ is the required map. It remains to show the converse.

   Suppose we have $\psi\colon N \to M$ such that $\psi\varphi = \mathrm{Id}_M$. We must first show that $N = \ker\psi + \varphi(M)$. Note that for any $x \in N$ we have that $x - \varphi(\psi(x)) \in \ker\psi$ and so in particular $x = (x - \varphi(\psi(x))) + \varphi(\psi(x)) \in \ker\psi + \varphi(M)$ and thus $N = \ker\psi + \varphi(M)$ as required.

   The result will now follow if we show that $\ker\psi \cap \varphi(M) = 0$. But this is clear, since if $\psi(\varphi(m)) = \mathrm{Id}_M(m) = 0$ then $m = 0$. ∎

3. Suppose that V is an irreducible kG-module and W is a 1-dimensional kG-module. Show that $V \otimes W$ is irreducible.

   Solution:
   Suppose that V is an arbitrary kG-module and $U \leq V$ a submodule of V. Then $U \otimes W \leq V \otimes W$ is a submodule: clearly it is a vector subspace, and for any $g \in G$, $u \in U$ and $w \in W$ we have $g(u \otimes w) = gu \otimes gw \in gU \otimes gW = U \otimes W$. This, however, is not sufficient. We must also check that all submodules of $V \otimes W$ occur in this manner. The dual W^* of W is then also 1-dimensional and irreducible, and we have that $W \otimes W^* \cong \operatorname{End} W \cong k$ by Schur's lemma. Thus we have that $(V \otimes W) \otimes W^* \cong V \otimes (W \otimes W^*) \cong V \otimes k \cong V$ and every submodule of $V \otimes W$ thus yields a submodule of V, but V is irreducible so it has no proper nontrivial submodules, thus the same is true for $V \otimes W$ and we are done.

Since the fact that $W \otimes W^* \cong \operatorname{End} W$ was not mentioned in the course, here is an alternate solution which does not use this: Suppose $X \leq V \otimes W$ is a nontrivial submodule. Then X is spanned by some vectors $x_1, \ldots, x_n \in V \otimes W$, but for any nonzero $w \in W$, each x_i is of the form $v_i \otimes w$ for some $v_i \in V$. Let Y denote the k-span of the vectors $v_1, \ldots, v_n$ in V. Clearly Y is a nontrivial vector subspace of V. Now, as vector spaces we have that $X = Y \otimes W$ by construction. Since X is a kG-submodule of $V \otimes W$ we have that $gX = X$, so for any $x = v \otimes w \in X$ we have that $gx = g(v \otimes w) = gv \otimes gw \in X$. But since $\dim W = 1$ we have that $gw = \lambda w$ for some $\lambda \in k$ and so $gv \otimes gw = gv \otimes \lambda w = \lambda(gv \otimes w) \in Y \otimes W$ and so $gY = Y$, thus Y is a nontrivial kG-submodule of V and so $Y = V$ and $X = W \otimes V$ is irreducible. ■

4. Let V be an irreducible kG-module.

   (a) Show that the multiplicity of V as a composition factor of the kG-module U is $\dim \operatorname{Hom}_G(\mathcal{P}(V), U)$.

   Solution:
   We prove this by induction on the Loewy length (recall that this is the name given to the common radical and socle length of U) of U. If U is semisimple and $\varphi\colon \mathcal{P}(V) \to U$ then $\operatorname{rad}(\mathcal{P}(V)) \leq \ker \varphi$ and so φ induces a map from $\mathcal{P}(V)/(\operatorname{rad}(\mathcal{P}(V)) \cong \operatorname{head}(\mathcal{P}(V)) \cong V$ to U and in particular we have that $\dim \operatorname{Hom}_G(\mathcal{P}(V), U) = \dim \operatorname{Hom}_G(V, U)$ is the multiplicity of V as a composition factor of U, as required. Now suppose that the desired result holds for any module U of Loewy length n and suppose that U has Loewy length $n+1$. Then the multiplicity of V as a composition factor of U is equal to the sum of its multiplicity as a composition factor of $\operatorname{rad} U$ and its multiplicity as a composition factor of $\operatorname{head} U$. As $\operatorname{rad} U$ has Loewy length n, we have that the multiplicity of V as a composition factor of $\operatorname{rad} U$ is $\dim \operatorname{Hom}_G(\mathcal{P}(V), \operatorname{rad} U)$, and since $\operatorname{head} U$ is semisimple we also have that the multiplicity of V as a composition factor of $\operatorname{head} U$ is $\dim \operatorname{Hom}_G(\mathcal{P}(V), \operatorname{head} U)$. So the multiplicity of V as a composition factor of U is $\dim \operatorname{Hom}_G(\mathcal{P}(V), \operatorname{rad} U) + \dim \operatorname{Hom}_G(\mathcal{P}(V), \operatorname{head} U)$. Now, it remains to show that $\dim \operatorname{Hom}_G(\mathcal{P}(V), U) = \dim \operatorname{Hom}_G(\mathcal{P}(V), \operatorname{rad} U) + \dim \operatorname{Hom}_G(\mathcal{P}(V), \operatorname{head} U)$. This follows immediately from the exactness of the functor $\operatorname{Hom}_G(P, -)$ for P any projective module, which we prove below. (Note one needs no knowledge of functors to make this observation)

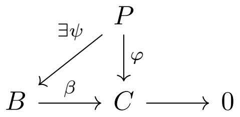
   Let P be a kG-module and $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence. For a kG-homomorphism $\varphi\colon M \to N$ let $\operatorname{Hom}_G(P, \varphi)\colon \operatorname{Hom}_G(P, M) \to \operatorname{Hom}_G(P, N)$ be given by $\operatorname{Hom}_G(P, \varphi)\colon \psi \mapsto \varphi\psi\colon P \to N$. Then the sequence

   $$0 \to \operatorname{Hom}_G(P, A) \xrightarrow{\operatorname{Hom}_G(P,\alpha)} \operatorname{Hom}_G(P, B) \xrightarrow{\operatorname{Hom}_G(P,\beta)} \operatorname{Hom}_G(P, C)$$

   is exact, and if P is projective then the induced map $\operatorname{Hom}_G(P, \beta)$ is surjective so that the sequence

   $$0 \to \operatorname{Hom}_G(P, A) \xrightarrow{\operatorname{Hom}_G(P,\alpha)} \operatorname{Hom}_G(P, B) \xrightarrow{\operatorname{Hom}_G(P,\beta)} \operatorname{Hom}_G(P, C) \to 0$$

   is exact. To show the first claim, it is sufficient to show that $\operatorname{Hom}_G(P, \alpha)$ is injective and $\operatorname{Hom}_G(P, \alpha) \circ \operatorname{Hom}_G(P, \beta)$ is zero, yet these are both clear since α is injective and $\alpha \circ \beta = \beta\alpha = 0$. For the latter claim, we require that $\operatorname{Hom}_G(P, \beta)$ is also surjective. Since β is surjective, for any $\varphi \in \operatorname{Hom}_G(P, C)$ we have



and so since P is projective there exists some map $\psi\colon P \to B$ such that $\varphi = \beta\psi = \mathrm{Hom}_G(P, \beta)(\psi)$ as required. ■

(b) Suppose also that $W \in \mathrm{Irr}_k\, G$. Deduce that the multiplicity of V as a composition factor of $\mathcal{P}(W)$ is the same as the multiplicity of W as a composition factor of $\mathcal{P}(V)$.

Solution:
By the previous part, the multiplicity of V as a composition factor of $\mathcal{P}(W)$ is equal to $\dim \mathrm{Hom}(\mathcal{P}(V), \mathcal{P}(W))$. Dualising the previous argument, since $\mathcal{P}(V)$ is also the indecomposable *injective* module with socle V, we have that the multiplicity of V as a composition factor of $\mathcal{P}(W)$ is also $\dim \mathrm{Hom}(\mathcal{P}(W), \mathcal{P}(V))$. But again by the previous part this is the multiplicity of W as a composition factor of $\mathcal{P}(V)$, as required. ■

5. Suppose that V is a relatively H-projective kG-module. Show that $V \otimes U$ is relatively H-projective for any kG-module U.

Solution:
Since V is relatively H-projective, V is a summand of a relatively H-free module, *i.e.* $V \mid \mathrm{Ind}_H^G W$ for some H-module W. Then

$$V \otimes U \mid \mathrm{Ind}_H^G(W) \otimes U \cong \mathrm{Ind}_H^G(W \otimes U_H)$$

by Lemma 4.7. But then $\mathrm{Ind}_H^G(W \otimes U_H)$ is relatively H-free and so $V \otimes U$ is relatively H-projective. ■

6. Let $P \in \mathrm{Syl}_p\, G$ be cyclic and normal in G and let $W := \mathrm{rad}\,\mathcal{P}(k)/\mathrm{rad}^2\,\mathcal{P}(k)$, where k denotes the trivial kG-module. Show that U, $V \in \mathrm{Irr}_k\, G$ lie in the same block if and only if $V \cong U \otimes W^{\otimes n}$ for some n.

Note: When initially released, this question was missing the (rather important) word *cyclic* and thus was impossible to prove due to not being true.

Solution:
As we saw in Lemma 4.29, the PIMs for G are uniserial of shape $[X \mid X \otimes W \mid X \otimes W^{\otimes 2} \mid \ldots \mid X \otimes W^{\otimes x}]$ for some integer x, irreducible kG-module X and 1-dimensional kG-module W. Since $\mathrm{head}\,\mathcal{P}(X) \cong \mathrm{soc}\,\mathcal{P}(X) \cong X$ we have that $X \cong X \otimes W^{\otimes x}$ and thus $W^{\otimes x} \cong k$. So U, $V \in \mathrm{Irr}_k\, G$ are composition factors of the same projective indecomposable kG-module if and only if $V \cong U \otimes W^{\otimes n}$ for some integer n. By Proposition 5.3 ii), the irreducible modules which lie in the same block as V are precisely $\{V \otimes W^{\otimes n} \mid n \in \mathbb{N}\}$ and the result follows since $W^{\otimes x} \cong k$. ■

7. (a) Prove that the blocks of $\mathrm{SL}_2(p)$ for $p > 2$ are as stated in Example 5.5.
   (b) Show that the Brauer trees for these blocks are as stated in Example 6.8 for $p > 2$.

Solution:
Let $G \coloneqq \mathrm{SL}_2(p)$. Throughout this question one should recall the definition of the irreducible kG-modules from Example 3.13, the fact that V_p is projective (shown in Example 3.38) and the non-split extensions between irreducible kG-modules constructed in Lemmas 4.32 and 4.33 within Example 4.31.

(a) We first note that V_p is projective and, since any extension of a nonzero kG-module by a projective kG-module must split, we have that V_p is the only irreducible module lying in its block (and this is the same for any projective irreducible kG-module for an arbitrary finite group G.

We begin with V_1 and determine the constituents of the block in which it lies. Taking $i = 1$ in Lemma 4.32, there is a non-split extension between V_1 and V_{p-2} and thus by Proposition 5.3 they lie in the same block. Next, taking $i = 3$ in Lemma 4.33 we note a non-split extension between V_{p-2} and V_3 and so V_3 also lies in the same block as V_1 and V_{p-2}. Note that upon iterating this process the parity of i never changes and so we pick up every irreducible module whose index is odd.

Next, taking $i = 2$ in Lemma 4.32 we observe that V_2 lies in the same block as V_{p-3} and taking $i = 4$ in Lemma 4.33 this block also contains V_4. Once again iterating this process we collect all irreducible modules indexed by even i. It remains only to show that these two collections of modules do not form a single block. But this is clear from the description of the projective indecomposable modules given at the end of Example 3.38, as all projective indecomposable modules only contain irreducible modules with indices of the same parity.

(b) Let $1 < i < p - 1$. Then by Example 3.38, $\mathcal{H}(\mathcal{P}(V_i)) \cong V_{p-1-n} \oplus V_{p+1-n}$ and thus, provided $p \neq (p \pm 1)/2$, the edge labelled by V_i shares a vertex with the edges labelled by V_{p-1-n} and V_{p+1-n}. Now suppose that $\varepsilon = \pm 1$, $p \equiv \varepsilon \mod 4$ and $i = (p \pm \varepsilon)/2$. Then $p \pm \varepsilon - (p \pm \varepsilon)/2 = (p \pm \varepsilon)/2$ and so the edge labelled by V_i shares a vertex only with the edge labelled by $V_{(p+3\varepsilon)/2}$ and its heart contains a module isomorphic to its socle as a composition factor with multiplicity one, confirming that it is the exceptional vertex with exceptionality two.

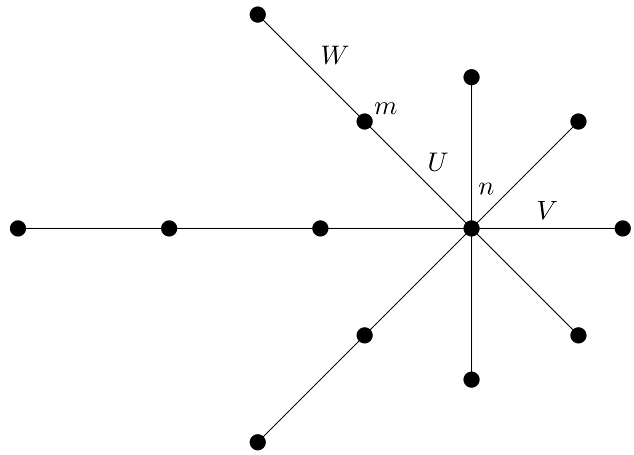
Finally, referring once again to Example 3.38 we see that V_1 shares a vertex with the edge labelled by V_{p-2} and V_{p-1} shares a vertex with the edge labelled by V_2. As such, the Brauer tree is a line as drawn with the indicated exceptional vertices and exceptionalities.

■

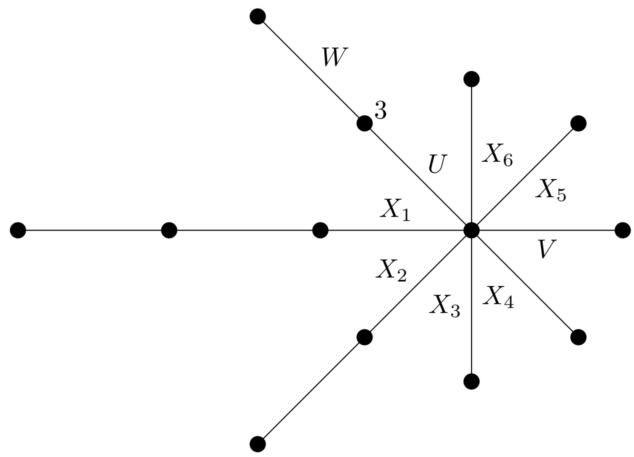
8. Prove Corollary 5.20: Let G be a finite group and B a block of kG. Then B is a simple algebra if and only if B has defect zero.

Solution:
Suppose that B has defect zero, so that the defect group of B is 1. Then by Theorem 5.9 every indecomposable B-module has trivial vertex and is thus projective. Thus B is semisimple, but B is indecomposable and thus is simple. Conversely, if B is simple then every B-module is projective (since B is semisimple as a B-module) and so Corollary 5.19 yields that the defect group of B is trivial. ■

9. Let B be a block with the below Brauer tree. Determine the structure of the projective indecomposable modules corresponding to the irreducible modules U, V and W in the cases $(m, n) = (3, 1)$ and $(m, n) = (1, 3)$.



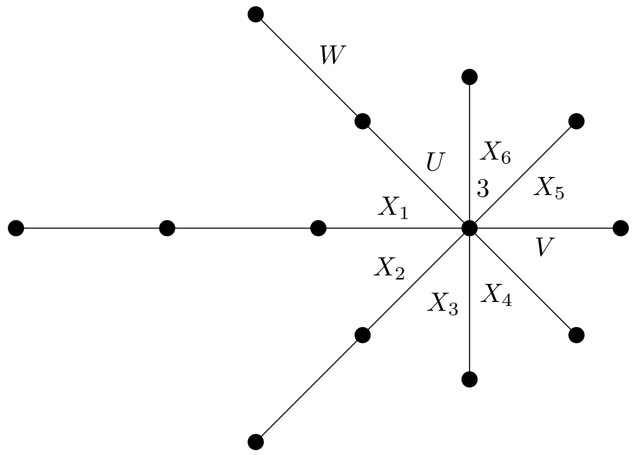
Solution:
This is a simple exercise in applying the algorithm described in Definition 6.3. The first thing one must do is apply labels to some of the edges not labelled in the question, which we shall do for each case. First up, we assume that $(m, n) = (3, 1)$.



First, we let $Y \sim [X_1 \mid X_2 \mid X_3 \mid X_4 \mid V \mid X_5 \mid X_6]$ denote the unique uniserial module with the indicated radical factors (starting from head $Y \cong X_1$) and let $Z \sim [X_5 \mid X_6 \mid U \mid X_1 \mid X_2 \mid X_3 \mid X_4]$ similarly denote the unique uniserial module with the indicated radical factors. Then in this case, the projective indecomposable modules may be described as follows with the heart of the module being depicted as a direct sum of two (possibly zero) uniserial modules.

$$\mathcal{P}(U) \sim \begin{array}{ccc} & U & \\ W & & \\ U & & \\ W & \bigoplus & Y \\ U & & \\ W & & \\ & U & \end{array} \qquad \mathcal{P}(V) \sim \begin{array}{c} V \\ Z \bigoplus 0 \\ V \end{array} \qquad \mathcal{P}(W) \sim \begin{array}{ccc} & W & \\ & & U \\ & & W \\ 0 & \bigoplus & U \\ & & W \\ & & U \\ & W & \end{array}$$

This leaves the case $(m, n) = (1, 3)$, illustrated below.



In this case, with Y and Z as described above, we have the following description of the projective indecomposable modules.

$$\mathcal{P}(U) \sim \begin{array}{ccc} & U & \\ & & Y \\ & & U \\ W & \bigoplus & Y \\ & & U \\ & & Y \\ & U & \end{array} \qquad \mathcal{P}(V) \sim \begin{array}{ccc} & V & \\ Z & & \\ V & & \\ Z & \bigoplus & 0 \\ V & & \\ Z & & \\ & V & \end{array} \qquad \mathcal{P}(W) \sim \begin{array}{c} W \\ U \\ W \end{array}$$

■