Modular Representation Theory

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1 Introduction

Let G be a finite group and k a field. A *representation* of G over k is simply a homomorphism from G to $\mathrm{GL}_n(k)$ for some positive integer n, and via this homomorphism we obtain an action of G on the natural vector space of $\mathrm{GL}_n(k)$. A simple definition yields a remarkably rich theory with a lot of structure to investigate. A typical first course in representation theory will focus on the case where k is algebraically closed and of characteristic zero, in particular $k = \mathbb{C}$. It is common, then, to focus on *character theory*, which associates to each representation of G a function $\chi\colon G \to \mathbb{C}$ which is invariant on conjugacy classes of G and is in fact sufficient to determine the isomorphism type of the representation as a whole. It is possible in this scenario to work with representations of a group G without ever even dealing with the fact that one has an action on a vector space, and indeed doing so can yield an abundance of information about both the group and its representations.

When k is an arbitrary field of characteristic $p > 0$ with $p \mid |G|$, however, character theory is somewhat weaker. The *Brauer characters* of G over k still tell us about some of the structure of its representations, but the information is no longer complete: there exist representations of G whose Brauer characters are the same but which are not isomorphic. Character theory still remains a powerful tool in this case, but we often require more information to get what we want. This is where *modular* representation theory comes in, and as the name suggests the focus is typically more on the structure of representations of G over k as *kG-modules*, or k–vector spaces equipped with an action of G. Through investigating the structure of the group algebra kG and its modules, we can again find an abundance of information about the group and its representations.

The goal of this course is to provide a brief introduction to the representation theory of a finite group over an algebraically closed field of characteristic $p \mid |G|$ and give a taste of some of the methods and structures associated with this work. We largely follow Alperin [1] for this treatment, which is (by the author's admission) already a book which attempts to go very deep into the subject without spending too much time broadening its approach. As such, we shall see a little bit of a lot of things, but anyone keen to work in this area will absolutely need to read some more thorough resources. Some other standard texts on representation theory are due to Navarro [14] and Isaacs [11] with an introduction by James–Liebeck commonly used for undergraduate courses [12]. For a contemporary overview if one wishes to find out what is currently known, there is Craven's guidebook [3] and if one simply wishes for a reference text then one is likely to find any classical results somewhere in the roughly 2500 pages written by Curtis and Reiner [5, 6, 7].

Throughout this course, we shall provide a very rapid introduction to the theory of A-modules for a finite-dimensional algebra A and specialise this to the case of group algebras kG for G a finite group. We shall introduce the concepts of free and projective modules, prove fundamental results such as Maschke's Theorem and investigate the structure of various projective and indecomposable modules for our group G. Further into the course, we shall begin to relate the structure of kG-modules to local subgroups of G (that is, normalisers of p-subgroups) and discuss vertices and sources and prove the Green Correspondence. We then take a brief foray into the theory of blocks, defect groups and their properties. Finally, we conclude the course with an investigation into what happens when the defect group of a block (or simply the Sylow p-subgroups of the group G) are cyclic and show how a structure known as a *Brauer tree* can be used to fully determine the structure of the projective indecomposable modules for G, and in fact all indecomposable modules.

Throughout the course, we revisit the example of $\mathrm{SL}_2(p)$ to determine all of its irreducible modules, then its projective indecomposable modules and their distribution into blocks.

2 Modules

In this section, we introduce the concept of modules and provide some of the fundamental results which we will require throughout the rest of the course. This is considered background material and so proofs will typically not be provided and for a more thorough treatment of these concepts the reader is referred to the references listed in this section or any other introductory text on rings/algebras and modules. All of the material from this section (and a lot more) should lie somewhere in the introductory chapter of [5], mostly within sections 1–3 and 5. Much (but not quite all) of this information may also be found in the first two chapters of [1].

Definition 2.1
Let R be a ring. A *left R-module* is an additive abelian group M with an operation $R \times M \to M$ with $(r, m) \mapsto rm$ such that, for all r, $s \in R$, m, $n \in M$, the following hold.

- $(r + s)m = rm + sm$ and $r(m + n) = rm + rn$.
- $(rs)m = r(sm)$.
- $1_R m = m$.

The final condition is not always required in all definitions.

Remark
The inclusion of the word *left* in the above definition suggests, correctly, that there is also a notion of *right* R-modules in which one instead has an action $M \times R \to M$ with $(m, r) \mapsto mr$ and adjusts the above conditions accordingly. In this course we are likely to only use left modules. It is not uncommon to see the notation ${}_R M$ to denote M as a left R-module or M_R to denote M as a right R-module.

Definition 2.2
Let A be a ring and R a commutative ring. We say that A is an *algebra* over R (or R-*algebra*) if there exists a homomorphism $\psi\colon R \to Z(R)$ such that $\psi(1_R) = 1_A$.

Remark
With this definition, every ring is a $\mathbb{Z}$-algebra.

In this course, by an algebra we will mean a *unital*, *associative* algebra; nonassociative algebras and non-unital algebras are also very important and frequently studied (*e.g.* Lie algebras) but are well outside the scope of this course. For simplicity we will also take R to be a field k and assume that A is finite-dimensional over k (a k-algebra is simply a vector space over k, so this makes sense). For a more general treatment of this, see the introductory chapter in [5]. In particular, we will mostly be concerned with algebras of the following type.

Definition 2.3
Let G be a group and k a field. The *group algebra* kG is the vector space over k with basis G and multiplication inherited from G extended linearly.

So kG is the set $\{\sum_{g \in G} a_g g \mid a_g \in k\}$ with the obvious addition and multiplication operations. Since our real goal in this course is to investigate the structure of kG and its modules, we will state the rest of the results in this section for algebras rather than rings.

Notation
For the remainder of this section, A will be an (associative, unital) algebra over an algebraically closed field k of characteristic $p \geq 0$ and M a left A-module. All algebras and modules in this course are assumed to be finitely generated.

Lemma 2.4
Let M be an A-module. Then we have the following properties for all $m \in M$.

- $0_A m = 0_M$ *(and so we may write* 0 *to mean either* 0_A *or* 0_M *with impunity).*
- $(-1)m = -m$.

Now that we have defined a new algebraic structure (thing), we go through the usual motions of defining subthings, factor things, thing homomorphisms and providing the thing isomorphism theorems. Even if you've never seen these for modules before, you almost certainly already know what they look like.

Definition 2.5
Given a left A-module M, a (left) A-submodule of M is a subset of M which is itself a left A-module under the operations inherited from M. If N is a submodule of M we write $N \leq M$. If the only submodules of M are 0 and M, we say that M is *simple*.

As with most algebraic structures, this boils down to the usual sub*thing* criterion:

Lemma 2.6
Let M be a left A-module. A subset $N \subseteq M$ is a submodule of M if and only if $N \neq \varnothing$, $N \leq M$ (N is an additive subgroup of M) and $N = AN := \{a * n \mid a \in A,\ n \in N\}$.

Definition 2.7
Let M, N be left R-modules. An *A-module homomorphism* (or *A-homomorphism*) from M to N is a map $\varphi\colon M \to N$ such that, for all $a \in A$, m, $n \in M$, $\varphi(m+n) = \varphi(m) + \varphi(n)$ and

$\varphi(am) = a\varphi(m)$. If φ is bijective then we call it an *isomorphism* of A-modules. We denote the set of such homomorphisms by $\mathrm{Hom}_A(M, N)$ and if $M = N$ we call φ an *endomorphism* and denote the set of such maps by $\mathrm{End}_A M$. We will often drop the subscript A from Hom and End when it is not strictly required. Finally, if $\varphi \in \mathrm{End}_A M$ is an isomorphism then we call φ an automorphism.

Lemma 2.8
Let M, N be left A-modules and $\varphi \colon M \to N$ be a homomorphism of A-modules. Then $\ker \varphi$ is a submodule of M and $\operatorname{im} \varphi = \varphi(M)$ is a submodule of N.

Lemma 2.9 (Schur's Lemma)
Let M, N be simple A-modules. Then either $\mathrm{Hom}_A(M, N) = 0$ or $M \cong N$ and, as vector spaces, $\mathrm{Hom}_A(M, N) = \mathrm{End}\, M \cong k$. In particular, the only endomorphisms of M are scalar multiples of the identity.

Schur's Lemma as stated above does not hold in the case where k is not algebraically closed. We shall see a little bit of what happens when k is not algebraically closed when we get to talking about group representations.

Definition 2.10
Let M, N be left A-modules with $N \subseteq M$. Then the factor group M/N is a left A-module under the operation $a(m + N) = am + N$ for $m + N \in M/N$, $a \in A$.

As with groups, we have the notion of a *composition series* for A-modules.

Definition 2.11
Let M be an A-module. A *composition series* for M is a sequence of submodules $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ and for each i we have that M_{i+1}/M_i is a simple A-module. These simple quotients are known as the *composition factors* of M.

In general, a composition series for a module need not exist, but in our case we assume that all modules are finitely generated and that our algebra A is finite-dimensional over k. This is enough to assume that our algebra A is an *Artinian ring* and thus all finite-dimensional A-modules have composition series. We conclude this brief aside with a theorem that likely looks very familiar.

Theorem 2.12 (Jordan–Hölder)
Suppose that M is a module with two composition series. Then the number of factors in these series are equal and, up to reordering, the factors are isomorphic as A-modules.

Let M be a simple A-module. Since for any nonzero $m \in M$ we have that $M = \{a * m \mid a \in A\}$, we see that M is in fact a quotient of A as an A-module. Since A is finitely generated, this tells us that in fact there are only finitely many simple A-modules up to isomorphism

Now it's time for the thing isomorphism theorems. We begin with a little setup. As is tradition, if there exists an isomorphism $\varphi \colon M \to N$ for (left) A-modules M and N we say that M and N are isomorphic and denote this by $M \cong N$. A useful tool in the proof of one of the isomorphism theorems is the below correspondence theorem for modules, which likely also looks like one you've seen before.

Theorem 2.13 (Correspondence theorem for modules)
Let M be a left A-module and $N \subseteq M$ a submodule. Then every submodule of M/N has the

form K/N for some submodule K of M containing N and there is an inclusion-preserving 1–1 correspondence ($K \leftrightarrow K/N$) between the submodules K of M containing N and the submodules of M/N. If $M \cong N$ then any isomorphism provides a 1–1 correspondence between the submodules of M and of N.

The below theorems are commonly referred to as the First, Second and Third isomorphism theorems for modules, respectively.

Theorem 2.14 (Isomorphism theorems for modules)
Let M, N be left A-modules. Then

1. *If $\varphi\colon M \to N$ is an A-module homomorphism then $M/\ker\varphi \cong \operatorname{im}\varphi$.*
2. *If L, $N \leq M$ are A-submodules then*
$$\frac{L+N}{N} \cong \frac{L}{L \cap N}.$$
3. *If N and L are submodules of M with $N \subseteq L$ then*
$$\frac{M/N}{L/N} \cong M/L.$$

Definition 2.15
Let M and N be A-modules. We define the (external) *direct sum* to be $M \oplus N \coloneqq \{(m,n) \mid m \in M,\ n \in N\}$. More generally, let $\{M_i \mid i \in I\}$ be a family of A-modules. Then $\bigoplus_{i\in I} M_i$ is the set of all $(m_i \mid i \in I)$ such that only finitely many m_i are nonzero. If the indexing set I is finite then this is equivalent to the direct product. Addition is applied componentwise and for $a \in A$ we have $a(m_i \mid i \in I) = (am_i \mid i \in I)$. The factors M_i are called *direct summands.*

Now instead let M, N be submodules of some A-module L such that $L = M + N \coloneqq \{m+n \mid m \in M,\ n \in N\}$ and each element $l \in L$ may be expressed uniquely as a sum $l = m + n$ for $m \in M$, $n \in N$. We then say that L is the (internal) *direct sum* of M and N and write $L = M \oplus N$.

The uniqueness condition in the above definition of internal direct sum is, thankfully, equivalent to checking that $M \cap N = 0$. The above two notions of direct sum are also equivalent, so we may use the symbol $\oplus$ without having to worry whether the sum is internal or not. For repeated direct sums of a module, we use the notation $M^{\oplus n} \coloneqq \bigoplus_{i=1}^{n} M$.

Definition 2.16
We say that a left A-module M is *decomposable* if there exist proper nontrivial submodules L, $N \subseteq M$ such that $M = L \oplus N$. If no such submodules exist, M is *indecomposable.*

Proposition 2.17
Let M be an A-module. Then M is a direct sum of simple A-modules if and only if every submodule of M is a direct summand.

Modules satisfying the above equivalent properties are called *semisimple.*

Definition 2.18
Let M be an A-module. If for every submodule $N \subseteq M$ there exists a submodule $L \subseteq M$ such that $M = N \oplus L$ then we say that M is *completely reducible.*

Complete reducibility is also equivalent to semisimplicity, but the terms themselves do have different meanings and it can be helpful to know which one to refer to specifically.

Proposition 2.19
Let M be an A-module. Then M is completely reducible if and only if M is semisimple.

Theorem 2.20 (Krull-Schmidt)
Let M be an A-module such that $M = N_1 \oplus N_2 \oplus \cdots \oplus N_r$ and $M = L_1 \oplus L_2 \oplus \cdots \oplus L_s$ for integers r, s and all N_i, L_i simple. Then $r = s$ and, up to reordering, $N_i \cong L_i$ for all i.

Corollary 2.21
Let L, M, N be A-modules with $M \oplus N \cong M \oplus L$. Then $N \cong L$.

Semisimple modules are, in a sense, the easiest modules to deal with. They are a friendly direct sum of simple modules, and thus if you understand what happens to the simple modules (which is not necessarily an easy thing to do) you understand everything. We now briefly provide a few structural results on semisimple algebras which we will be able to apply to group representations later on.

Definition 2.22
Let A be an algebra. Then the *opposite algebra* A^o is the set A with the operation $a \cdot b := ba$.

Lemma 2.23
Let A be an algebra. Then $\operatorname{End} A \cong A^o$.

Lemma 2.24
If $M = M_1 \oplus M_2 \oplus \cdots \oplus M_t$ then $\operatorname{End} M$ *is the matrix algebra represented below.*

$$\begin{pmatrix} \operatorname{End} M_1 & \operatorname{Hom}(M_1, M_2) & \cdots & \operatorname{Hom}(M_1, M_t) \\ \operatorname{Hom}(M_2, M_1) & \operatorname{End} M_2 & \cdots & \operatorname{Hom}(M_2, M_t) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}(M_t, M_1) & \operatorname{Hom}(M_t, M_2) & \cdots & \operatorname{End} M_t \end{pmatrix}$$

Theorem 2.25
If A is a simple algebra then A is a matrix algebra.

Sketch proof. Let S be a simple submodule of A and let M denote the sum of all submodules of A isomorphic to S. Then $\operatorname{End} M \cong \mathrm{M}_n(k)$ where n is the number of summands in M. One then shows that $A = M$ and the result follows by Lemma 2.23. ■

An important thing to note above is that the integer n is the number of times that S appears as a submodule of A, and is also the dimension of S. Now, the following theorem is a fundamental result regarding semisimple algebras. Note, the statement (and name!) of the below theorem will vary dependent on how one defines an algebra. In our case, since we always assume that algebras are finite-dimensional over k we have that our algebras are always Artinian and so Wedderburn's Theorem will suffice. If we do not require that our algebras be finite-dimensional then we would instead require the Artin–Wedderburn Theorem.

Theorem 2.26 (Wedderburn)
Let A be a semisimple algebra. Then A is the direct sum of matrix algebras.

Combined with the observation before the theorem, this yields the following corollary.

Corollary 2.27
Let A be a semisimple algebra and S be a simple A-module. Then the number of times that S appears as a submodule of A as an A-module is equal to the dimension of S.

Obviously, not every module is semisimple, so let's take a look at one that isn't.

Example 2.28
Let $G \coloneqq C_2 \times C_2 = \langle x, y \rangle$ and let $k = \mathbb{F}_2$ (for this choice of G this is equivalent to working over the algebraic closure but, mercifully, finite). Recall Definition 2.3: the group algebra kG is a module over itself with the obvious action. It has dimension 4 (over k), and as we shall see is not a semisimple kG-module. Let $\sigma = \sum_{g \in G} g = 1 + x + y + xy \in kG$. Then $k\sigma$ is a 1-dimensional subspace of kG. If you're bored, you can check that there are 15 3-dimensional subspaces of kG of which 8 are complements to $k\sigma$. None of these 8 subspaces are submodules of kG, though, so kG is not semisimple.

Now, since G is a group, we have a rather uninteresting homomorphism $\hat{\varphi} \colon G \to 1$ onto the trivial group. Extend this map linearly to kG and we get a kG-homomorphism $\varphi \colon kG \to k$ called the *augmentation map*. Since φ is a homomorphism, $\ker \varphi = \Delta(G)$ is a kG-submodule of kG. In this case, since $\operatorname{char} k = 2$, $\Delta(G)$ is the 3-dimensional subspace of kG spanned by σ, $1 + x$ and $1 + y$. Thus we see that $kG/\Delta(G)$ is a 1-dimensional kG-module which, as before, does not appear as a submodule of kG. One may also go further and check that $\Delta(G)/k\sigma$ is a 2-dimensional semisimple module M (to check that it is semisimple it is sufficient to note that $am = m$ for all $a \in kG$, $m \in M$).

In the above example, we have explored the structure of a module but we lack the terms necessary to describe what we have uncovered. We should probably fix that.

Definition 2.29
Let A be an algebra. The *radical* $\operatorname{rad} A$ of A is the smallest submodule of A such that $A/\operatorname{rad} A$ is semisimple.

Equivalently, $\operatorname{rad} A$ is the intersection of all maximal submodules of A or the largest nilpotent ideal of A. A noteworthy fact here is that if A is any algebra then $A/\operatorname{rad} A$ is a semisimple algebra.

Definition 2.30
An algebra A is said to be *local* if $A/\operatorname{rad} A \cong k$.

Lemma 2.31
The algebra A is local if and only if every element of A is either nilpotent or invertible.

Theorem 2.32
An A-module M is indecomposable if and only if $\operatorname{End} M$ *is local.*

Proposition 2.33
If M is an A-module then the following are equal.

- $(\operatorname{rad} A)M$.

- *The smallest submodule of M with semisimple quotient.*
- *The intersection of all maximal submodules of M.*

We denote the submodule of M given above by $\operatorname{rad} M$, with the corresponding semisimple quotient $\operatorname{head} M := M/\operatorname{rad} M$ called the *head* of M.

We may then use this to define the *radical series* of M: Let $\operatorname{rad}^0 M := M$ and then define $\operatorname{rad}^{i+1} M := \operatorname{rad}(\operatorname{rad}^i M)$ for each i. Since the module M is finitely generated, $\operatorname{rad}^r M = 0$ for some integer $r \geq 0$ which we call the *radical length* of M. We then have that for each i, $\operatorname{rad}^i M/\operatorname{rad}^{i+1} M$ is a semisimple module and we call these modules the *radical factors* of M. This gives a description of M in terms of semisimple modules. The first radical factor, $\operatorname{head} M$, is sometimes also referred to as the *top* of M, and looking at radical factors of a module is sometimes regarded as starting at the 'top' of a module and working 'down.' One may also start at the 'bottom' of a module and work up, as follows.

Let $\operatorname{soc} M$ be the sum of all simple submodules of M, let $\operatorname{soc}^0 M := 0$ and define the *socle series* of M by taking $\operatorname{soc}^{i+1}(M)$ to be the preimage in M of $\operatorname{soc}(M/\operatorname{soc}^i M)$. Then as before we have that $\operatorname{soc}^{i+1} M/\operatorname{soc}^i M$ is a semisimple module and for some r we have that $\operatorname{soc}^r M = M$. We call this r the *socle length* of M.

Given an A-module M, the radical and socle lengths of M are equal and called the *Loewy length* of M.

Definition 2.34
Let M be an A-module. We define the *heart* of M to be the module $\mathcal{H}(M) := \operatorname{rad} M/(\operatorname{rad} M \cap \operatorname{soc} M)$.

Example 2.35
Referring back to Example 2.28, we showed that $\operatorname{soc} kG = k\sigma$, $\operatorname{rad} kG = \Delta(G)$, $\operatorname{head} kG$ is 1-dimensional (thus simple) and $\mathcal{H}(M) = \Delta(G)/k\sigma$ is a 2-dimensional semisimple module. In this case the socle and radical series yield the same semisimple factors, though this need not always be the case. Let k denote the simple kG-module with trivial action (so $am = m$ for all $a \in kG$, $m \in k$). Decomposing the module in this way yields the following picture of kG, with the head of the module at the top and socle at the bottom.

$$\begin{matrix} k \\ k \oplus k \\ k \end{matrix}$$

We shall see later that it is no accident that all the simple modules here are trivial.

Proposition 2.36
If M is an A-module then the following are equivalent:

- *M has a unique composition series.*
- *The radical factors of M are simple.*
- *The socle factors of M are simple.*

Any module satisfying the above equivalent properties is called *uniserial*.

We now finish off out background section on modules with a brief look at free modules and, most importantly for us, projective modules. A free module is essentially just a direct sum of copies of A as a module, but it is commonly formally defined via the following property.

Proposition 2.37
An A-module M is free if and only if there exists some subspace $X \subset M$ such that any linear transformation from X to any A-module N extends uniquely to a module homomorphism $M \to N$. The subspace X is called a basis *for M.*

While free modules are useful and important in general, our main motivation for defining them at this point is as a means to define projective modules.

Theorem 2.38
Let M, N and P be A-modules. Then the following are equivalent:

- *P is a direct summand of a free module.*
- *If $\varphi\colon M \twoheadrightarrow P$ is a homomorphism of A-modules then $\ker\varphi$ is a direct summand of M.*
- *If $\varphi\colon N \twoheadrightarrow M$ and $\psi\colon P \to M$ are homomorphisms of A-modules then there exists an A-module homomorphism $\rho\colon P \to N$ such that $\varphi\rho = \psi$.*

A module P satisfying any of the above properties is called *projective*. The third condition is commonly expressed in the form of a commutative diagram, but we need a brief definition first.

Definition 2.39
Let M_i be A-modules (possibly zero) for $i \in \mathbb{Z}$ with maps $\varphi_i\colon M_i \to M_{i+1}$. We say that the sequence

$$\cdots \xrightarrow{\varphi_{i-2}} M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \xrightarrow{\varphi_{i+1}} \cdots$$

is *exact* if for each i we have that $\ker\varphi_i = \operatorname{im}\varphi_{i-1}$.

The most common exact sequence we shall see is a *short exact sequence*, which is a sequence of the form

$$0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$$

where the exactness means that ι has kernel zero and is thus an injection and π is a surjection as the kernel of the (zero) map following it must be all of N. We say that such a sequence *splits* if $M \cong L \oplus N$ with ι and π corresponding to the natural inclusion and projection maps, respectively.

Now, the third condition of Theorem 2.38 is commonly expressed by saying that the following diagram commutes (with exact bottom row).

$$\begin{tikzcd} & P \arrow[dl, "\exists\rho"'] \arrow[d, "\psi"] & \\ N \arrow[r, "\varphi"] & L \arrow[r] & 0 \end{tikzcd}$$

We will be seeing projective modules often throughout the course as they are very important structures in representation theory.

Definition 2.40
Let M and N be modules. An *extension* of M by N is a module E and a short exact sequence

$$0 \to N \to E \to M \to 0.$$

Two such extensions (with modules E_1 and E_2 are said to be *equivalent* if there exists a map $\varphi \colon E_1 \to E_2$ (which is necessarily an isomorphism) such that the below diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{Id}"] & E_1 \arrow[r] \arrow[d, "\varphi"] & M \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & N \arrow[r] & E_2 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

We say that an extension *splits* if it is equivalent to the extension

$$0 \to N \xrightarrow{\iota} N \oplus M \xrightarrow{\pi} M \to 0$$

with ι and π the natural inclusion and projection maps, respectively. We denote the set of extensions of M by N by $\mathrm{Ext}^1_A(M, N)$.

There is also a generalisation of extensions called *n-extensions* in which we replace the module E by a sequence of modules $E_1 \to \cdots \to E_n$, and in this framework an extension is simply a 1-extension. We will not be discussing n-extensions here, but ordinary extensions may arise in future.

3 Group representations

We now define our main object of study in this course: a *representation* of a finite group G. Many of the results stated from here on out will still hold in the case where G is infinite, but much of our machinery relies on the fact that the group algebra (and thus any irreducible representation) is a finite dimensional k–vector space.

Let G be a finite group and k an algebraically closed field of characteristic $p \geq 0$. We have already seen the group algebra kG in Definition 2.3. With our definition of representation, we shall now see why it is important.

Definition 3.1
Given a finite group G and a field k, a *representation* of G is a homomorphism $\rho \colon G \to \mathrm{GL}(V)$ for some k–vector space V. This yields a group action of G on the vector space V and also gives V the structure of a left kG-module. We say that a kG-module V is *faithful* if the only element of G which acts trivially on V is the identity element.

Throughout this course, we will typically prefer to use the term *irreducible* over simple when talking about kG-modules.

Definition 3.2
Let V be a kG-module. The *G–fixed space* of V is $V^G := \{v \in V \mid gv = v \ \forall g \in G\}$. Any kG-module V with $V^G = V$ will be referred to as *trivial*, with the term *the trivial module* specifically referring to the 1-dimensional trivial module typically denoted k.

Example 3.3
The group algebra kG is of course a kG-module equipped with the obvious action. This is often called the *regular representation* of G and is a free kG-module. If G is not trivial then kG is not simple, though it can be indecomposable.

Example 3.4
Let G be a finite group and k a field. The subspace of kG spanned by the element $\sigma := \sum_{g \in G} g$ is a 1-dimensional (dimension here is taken over k) submodule of kG called the *trivial kG-module*. We often denote this module by k.

Example 3.5
Let G be a finite group. $H \leq G$ and k a field. The *permutation module* of G on H is the vector space $k(G/H)$ spanned by the cosets of H in G with the obvious G-action. Many important representations of groups may be obtained as sections (quotients of submodules) of permutation modules.

We'll see an important generalisation of permutation modules later on, but for now we lack the machinery to properly talk about them.

Example 3.6
Let $G = S_n$ and $H = \mathrm{Stab}_G(1) \cong S_{n-1} \leq G$. Then the permutation action of G on the cosets of H is isomorphic to its natural action on n points. The normal way to regard the permutation module on this subgroup then is to take a vector space V with basis $\{v_1, \dots, v_n\}$ and take the G-action on V to be $gv_i = v_{i^g}$ and extend this linearly to the whole space. There is again an obvious fixed space $V^G = \sum_{i=1}^n v_i$, and if $p > n$ we have that $W := V/V^G$ is irreducible. We call this module W the *deleted permutation module*.

Notation
For the remainder of this section, let G be a finite group and k be, unless otherwise stated, an algebraically closed field of characteristic $p \geq 0$. All kG-modules V are assumed to be finite-dimensional over k. If a kG-module V is simple, we will typically instead say that V is *irreducible*, and if it is not irreducible then it is *reducible*. Let $\mathrm{Irr}_k\, G$ denote the set of irreducible kG-modules, or $\mathrm{Irr}\, G$ if the field is clear.

We denote $\mathrm{Hom}_{kG}(V, W)$ by $\mathrm{Hom}_G(V, W)$ if it is necessary to specify the group but will often drop the G entirely.

We start with Maschke's Theorem, which handily splits the representation theory of a given finite group into two cases.

Theorem 3.7 (Maschke's Theorem)
Let G be a finite group and k an algebraically closed field of characteristic $p \geq 0$. Then the group algebra kG is semisimple if and only if $p \nmid |G|$.

Proof. Suppose that $p \mid |G|$. Then $p > 0$ is a prime. We have seen that the trivial module is a 1-dimensional submodule of kG and so, if kG were semisimple, we would have that kG is a sum of its simple submodules and in particular every composition factor of kG is a direct summand. Now, by Corollary 2.27 we have $\dim \mathrm{Hom}_G(k, kG) = 1$ (this is the number of times k appears as a submodule of kG) and thus we require that in any composition series of kG the module k appears as a factor only once.

Take the homomorphism of G onto the trivial group and extend this map linearly to a map $kG \twoheadrightarrow k$. The kernel of this map $\Delta(G)$ is a submodule of kG called the *augmentation ideal* of kG and is made up of all elements $\sum_{g\in G} \alpha_g g$ such that $\sum_{g\in G} \alpha_g = 0$. Furthermore, $kG/\Delta(G) \cong k$ since $\dim \Delta(G) = |G| - 1 = \dim kG - 1$ and $g \cdot 1 = 1 + (g-1) \in 1 + \Delta(G)$ for all $g \in G$, meaning that G has a fixed point on $kG/\Delta(G)$ and, since the module is 1-dimensional, G thus fixes the whole space.

Now, since $p \mid |G|$, we have that the obvious trivial submodule of G also lies in $\Delta(G)$. This yields a composition series $0 \subseteq k \subseteq \Delta(G) \subseteq kG$ with composition factors k, $\Delta(G)/k$ and k. As k appears twice in this composition series we see that kG is not semisimple.

Now, suppose that $p \nmid |G|$. Then $|G|$ is invertible in k. Let V be a kG-module and take $U \subseteq V$, it is sufficient to show that $V = U \oplus W$ for some submodule $W \subseteq V$. Let $\pi\colon V \twoheadrightarrow U$ be a projection map onto U (so $\pi|_U$ is the identity map). As vector spaces, we have that $V = U \oplus \ker \pi$. If we can show that π is a kG-homomorphism, then $\ker \pi$ will be a submodule of V and thus we will be done. It probably isn't, but we can fix that.

Let
$$\pi' := \frac{1}{|G|} \sum_{g\in G} \pi^g.$$
It is clear that π' is a linear map, and easy to check that $\pi'(V) \subseteq U$ and $\pi'|_U$ is the identity map. Let $h \in G$, $v \in V$. Then
$$\pi'(hv) = \frac{1}{|G|} \sum_{g\in G} \pi^g hv = \frac{1}{|G|} \sum_{g\in G} hh^{-1}g^{-1}\pi ghv = h\frac{1}{|G|} \sum_{x\in G} \pi^x v = h\pi'(v)$$
since hg runs over G as g does, as required. ∎

So, if $p \nmid |G|$ we have that kG is semisimple, and in fact this tells us that all kG-modules are semisimple. This is the case in ordinary representation theory, and if the field k is algebraically closed then this is more or less equivalent to working over $\mathbb{C}$. When $p \mid |G|$ things are vastly more complicated. We have already seen an example of this in Example 2.28, which is relevant to the next theorem.

Definition 3.8
We say that an element of G is *p-regular* if $p \nmid |g|$. The p-regular conjugacy classes of G are the conjugacy classes of p-regular elements.

Theorem 3.9
The number of simple kG-modules is equal to the number of p-regular conjugacy classes of G.

The proof of the above theorem is not short, and may be found in [1, pp. 17–20].

Corollary 3.10
If $\mathrm{char}\, k = 0$, then the number of irreducible kG-modules is equal to the number of conjugacy classes of G.

Corollary 3.11
If G is a p-group then there is only one simple kG-module. In particular, this module is the (simple) trivial module.

So as we saw in Example 2.28, every composition factor of kG when G is a p-group is in fact a trivial module. Of course, it also turned out that the socle and head of kG were both simple, this is also a property unique to p-groups but we shall see why later. We now deal with two more examples, one somewhat more complicated than the other.

Example 3.12
Let $G = \langle g \rangle$ be a cyclic group of order $n = p^a r$ for $p \nmid r$. Then the polynomial $x^r - 1$ is separable and thus has r distinct roots in k (This is why we take k to be algebraically closed — a smaller field will suffice, but $\mathbb{F}_p$ might not). Fix one such root λ. Form a 1-dimensional k–vector space where g^i acts as λ^i. Since $\lambda^n = (\lambda^r)^{p^a} = 1$ this tells us that this is in fact a kG-module and plainly simple with isomorphism type determined by the choice of λ. As the group G has r p-regular conjugacy classes, we in fact see that all irreducible kG-modules are of this form.

Note that since k has characteristic p we have $x^n - 1 = (x^r - 1)^{p^a}$ meaning that every nth root of unity is in fact an rth root of unity over k.

Example 3.13
Let $G = \mathrm{SL}_2(p)$ and let k have characteristic p. Then G has p conjugacy classes of p-regular elements, and thus $|\mathrm{Irr}_k\, G| = p$. As we will be revisiting this example, we should construct the p irreducible kG-modules.

Let V be the natural kG-module, regarded as the 2-dimensional space of column vectors with $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $gX = aX + cY$ and $gY = bX + dY$. Let $k[X, Y]$ be the polynomial ring in X and Y and note that the action just described can be used to define an action of G on the entire space. Let V_n denote the subspace of $k[X, Y]$ of homogeneous polynomials of degree $n - 1$. Then V_n is a kG-module with $V \cong V_2$ and $V_1 \cong k$. A basis for each V_n is $\{X^{n-1}, X^{n-2}Y, \ldots, XY^{n-2}, Y^{n-1}\}$, yielding $\dim V_n = n$.

We aim to show that $\mathrm{Irr}_k\, G = \{V_1, V_2, \ldots, V_p\}$. As there are p modules here, it is sufficient to show that they are all simple. Since it is 1-dimensional, it is clear that V_1 is simple. Now let $1 \leq n < p$ and let

$$g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $0 \leq i \leq n$, let W_{i+1} be the $i + 1$-dimensional vector subspace of V_{n+1} spanned by the basis $\{X^i Y^{n-i}, X^{i-1}Y^{n-i+1}, \ldots, XY^{n-1}, Y^n\}$ and set $W_0 := 0$. This gives us a chain of subspaces

$$0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_n \subsetneq W_{n+1} = V_{n+1}.$$

We now prove the following for $0 < i \leq n$ by induction:

i) W_i is a $k\langle g \rangle$-submodule of V_{n+1}.

ii) W_i/W_{i-1} is a trivial $k\langle g\rangle$-module.

iii) Each element of $W_i \setminus W_{i-1}$ generates W_i as a $k\langle g\rangle$-module.

Proof. For the case $i = 1$, note that $gX = X + Y$ and $gY = Y$, thus $gY^n = Y^n$ and so g acts trivially on W_1.

Now suppose that the claim holds for i. Taking the binomial expansion of $(X + Y)^i$, we have

$$gX^iY^{n-i} = (X+Y)^iY^{n-i} = X^iY^{n-i} + \begin{pmatrix} i \\ 1 \end{pmatrix} X^{i-1}Y^{n-i+1} + u$$

for $u \in W_{i-1}$. This shows that W_{i+1} is a $k\langle g\rangle$-submodule of V_{n+1} since clearly $gX^iY^{n-i} \in W_{i+1}$, any element of W_{i+1} may be written in the form $aX^iY^{n-i} + x$ for $x \in W_i$ and W_i is also a $k\langle g\rangle$-submodule of V_{n+1} by assumption. Since W_{i+1}/W_i is 1-dimensional and we see that $gX^iY^{n-i} - X^iY^{n-i} \in W_i$ we also have that W_{i+1}/W_i is a trivial $k\langle g\rangle$-module. Finally, since $i < p$, $\begin{pmatrix} i \\ 1 \end{pmatrix} \neq 0$ so that $(g-1)X^iY^{n-i} \in W_i \setminus W_{i-1}$. Take $v \in W_{i+1} \setminus W_i$, then as above we may express v as $aX^iY^{n-i} + x$ for $x \in W_i$ and $a \neq 0$ so that

$$(g-1)v = a(g-1)X^iY^{n-i} + (g-1)x.$$

Now, $(g-1)x \in W_{i-1}$ as W_i/W_{i-1} is a trivial module and so $(g-1)v \in W_i \setminus W_{i-1}$. By induction, the $k\langle g\rangle$-module generated by v therefore contains W_i since it contains an element of $W_i \setminus W_{i-1}$ and clearly also contains v, thus is the whole of W_{i+1} as required. ■

In particular, X^n generates V_{n+1} as a $k\langle g\rangle$-module. Now, $|g| = p$ and so $\langle g\rangle$ is a p-group and thus the only irreducible $k\langle g\rangle$-module is the trivial module by Corollary 3.11. Thus $\operatorname{soc} V_{n+1} = V_{n+1}^{\langle g\rangle}$ is spanned by Y^n. An identical proof then shows that Y^n generates V_{n+1} as a $k\langle h\rangle$-module and, as a $k\langle h\rangle$-module $\operatorname{soc} V_{n+1} = V_{n+1}^{\langle h\rangle}$ is spanned by X^n.

Now, suppose that $0 \neq W \leq V_{n+1}$. Then clearly W is a $k\langle g\rangle$-submodule of V_{n+1} and in particular must contain some irreducible $k\langle g\rangle$-submodule of V_{n+1}. But as a $k\langle g\rangle$-module $\operatorname{soc} V_{n+1}$ is irreducible and spanned by Y^n, thus W contains Y^n. But then W must contain the $k\langle h\rangle$-submodule of V_{n+1} generated by Y^n, which is V_{n+1}. So V_{n+1} is irreducible, as required.

The group $\mathrm{SL}_2(p)$ is an example we will return to multiple times throughout the course. For now though, we need to continue building up machinery, starting with Clifford's Theorem.

Definition 3.14
Let V be a kG-module and $H \leq G$. We denote the *restriction* of V to H by V_H. This is equivalent to restricting to the subalgebra kH of kG.

Theorem 3.15 (Clifford)
If V is a semisimple kG-module and $N \trianglelefteq G$ then V_N is semisimple.

Proof. It suffices to show this in the case where V is irreducible. Suppose that $W \leq V_N$. Then for any $g \in G$ we have that $gW \leq V_N$ as $n(gW) = gg^{-1}ngW = gn^gW = gW$ for any $n \in N$. Now let $U \leq V_N$ be an irreducible kN-submodule. Then $\sum_{g\in G} gU$ is a semisimple submodule of V_N, but this is clearly also a kG-module and thus is the entirety of V_N as required. ■

Clifford's theorem is a very important theorem in the representation theory of groups. So much so that the study of representations restricted to normal subgroups is typically referred to as 'Clifford Theory.' This is typically most useful when looking at soluble groups due to the abundance of normal subgroups.

We now return to our previous discussion of projective modules and specialise it a little further to the case of kG-modules.

While the group algebra kG need not be semisimple, it is usually decomposable. Indeed, the module kG is a free module and so its indecomposable direct summands are precisely the projective indecomposable kG-modules. We will refer to these modules as the *PIMs* for kG (or simply for G if the field is clear). This also yields the following corollary to Maschke's Theorem (Theorem 3.7):

Corollary 3.16
Let G be a finite group and k be a field whose characteristic does not divide $|G|$. Then every kG-module is projective.

Theorem 3.17
There is a one-to-one correspondence between isomorphism classes of projective indecomposable kG-modules and isomorphism classes of irreducible kG-modules given by $P \leftrightarrow P/\operatorname{rad} P$.

There are a number of important consequences of this theorem. First, we see that $\operatorname{head} P$ is irreducible and that each irreducible module is the head of some projective indecomposable module and second, if P and Q are both PIMs with isomorphic heads then $P \cong Q$. This also means that $\operatorname{rad} P$ is the unique maximal submodule of P. With this in mind, we have the following.

Definition 3.18
Let $V \in \operatorname{Irr}_k G$. The unique projective indecomposable module with V as a quotient is called the *projective cover* of V and we denote this cover by $\mathcal{P}(V)$.

Combining Theorem 3.17 with Corollary 2.27, we obtain the following.

Corollary 3.19
In a decomposition of the free module kG into a direct sum of indecomposable submodules, each isomorphism type of indecomposable projective module occurs as many times as the dimension of the corresponding irreducible module.

So while we need not have that the group ring kG is semisimple, we do have a decomposition

$$kG = \bigoplus_{V \in \operatorname{Irr}_k G} \mathcal{P}(V)^{\oplus \dim V}.$$

Lemma 3.20
Let V be a kG-module with $V/\operatorname{rad} V$ irreducible and isomorphic to $P/\operatorname{rad} P$ for some projective indecomposable module P. Then there exists a kG-module homomorphism $\varphi\colon P \twoheadrightarrow V$.

Theorem 3.21
If P is a projective kG-module and $H \leq G$ then P_H is a projective kH-module.

Proof. It is easy to check that $(kG)_H \cong (kH)^{\oplus [G:H]}$ and so in particular we have that kG is free as a kH-module. Thus any projective kG-module, being a direct summand of a free kG-module, restricts to a summand of a free kH-module and is thus projective for H. ∎

Corollary 3.22
If $P \in \operatorname{Syl}_p G$ has order p^a then every projective kG-module has dimension divisible by p^a.

Proof. By Corollaries 3.11 and 3.19, kP is indecomposable. Thus every projective kP-module is free and so in particular has dimension divisible by $\dim kP = |P| = p^a$. ■

Lemma 3.23
If $R \in \operatorname{Syl}_p G$ and $R \trianglelefteq G$ with M a kG-module then $\operatorname{rad} M = \operatorname{rad} M_R$. Moreover if $R = \langle x \rangle$ is cyclic then $\operatorname{rad} M = (1-x)M$.

Before we introduce any more concepts, we should use some of the ones we have to flesh out some of the previous examples. We first return to the case where G is cyclic.

Example 3.24
Let $G = \langle g \rangle$ be cyclic of order n and let V be a kG-module afforded by the representation $\varphi\colon G \to \mathrm{GL}(V)$. Then the eigenvalues of $\varphi(g)$ are nth roots of unity and in fact $V = V_1 \oplus \ldots \oplus V_s$ where each V_i is a Jordan block, that is, $\varphi(g)$ acts on V_i as the matrix

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

for some eigenvalue λ of $\varphi(g)$. Now, there is only one (one-dimensional) subspace of eigenvectors for $\varphi(g)$ in V_i, indicated by the bottom row of the matrix, and so V_i is indecomposable. As such, each indecomposable kG-module is a Jordan block for some nth root of unity λ. Assume now that $V = V_i$ is indecomposable with basis $v_1, \ldots, v_m$ corresponding to the above matrix. Let $n = p^a r$ as before with $p \nmid r$, so that λ is an rth root of unity as in Example 3.12. Let $\lambda_1, \ldots, \lambda_r$ denote the rth roots of unity. Then

$$\varphi(g)^r - \mathrm{Id}_V = (\varphi(g) - \lambda_1 \mathrm{Id}_V)(\varphi(g) - \lambda_2 \mathrm{Id}_V) \cdots (\varphi(g) - \lambda_r \mathrm{Id}_V)$$

and so $\varphi(g)^r - \mathrm{Id}_V = (\varphi(g) - \lambda \mathrm{Id}_V)S$ for some nonsingular linear transformation S of V which commutes with $\varphi(g)$. Further,

$$0 = \varphi(g)^n - \mathrm{Id}_V = (\varphi(g)^r - \mathrm{Id}_V)^{p^a} = (\varphi(g) - \lambda \mathrm{Id}_V)^{p^a} S^{p^a}$$

and so $(\varphi(g) - \lambda \mathrm{Id}_V)^{p^a} = 0$. But $(\varphi(g) - \lambda \mathrm{Id}_V)v_i = v_{i+1}$ for each $i < m$ with $(\varphi(g) - \lambda \mathrm{Id}_V)v_m = 0$ so that $m \leq p^a$. Thus there are n possible structures for V since there are r possible choices of a root of unity and $m = \dim V$ is at most p^a.

In this setup, the irreducible modules seen before are simply the 1-dimensional Jordan blocks. It is easy to see that the element $g^r - 1 \in kG$ is nilpotent and thus lies in $\operatorname{rad} kG$ since kG is a commutative algebra. Now, $(g^r - 1)V = \operatorname{rad} V$ is the subspace spanned by $v_2, \ldots, v_m$ and so $\operatorname{rad} V$ is the Jordan block of dimension $\dim V - 1$ with eigenvalue λ. One may iterate this to find that $\operatorname{rad}^i V$ shrinks by one dimension at a time, so that all radical layers have dimension one and in particular are irreducible. In fact, all radical layers are the same irreducible module: the one corresponding to λ. Let V_λ denote this irreducible module. Then each indecomposable module for kG is of the form

$$\begin{array}{c} V_\lambda \\ V_\lambda \\ V_\lambda \\ \vdots \\ V_\lambda \end{array}$$

for some rth root of unity λ, where the composition length of such a module is at most p^a. This shows that every indecomposable module for a cyclic group G is in fact uniserial with composition length at most $|G|_p$ (the p-part of $|G|$). Further, the indecomposable modules with composition length p^a are in fact the PIMs for kG.

The next example is a more complicated version of one we have seen before. In this next case, the indecomposable modules for G very much do not need to be uniserial.

Example 3.25
Let $G \coloneqq C_p \times C_p = \langle x, y\rangle$. For each $n \in \mathbb{N}$, let V_n be a $2n$-dimensional k–vector space with basis $v_1, \ldots, v_n, w_1, \ldots, w_n$. Let X be the linear transformation of V_n such that $Xv_i = w_i$ and $Xw_i = 0$ for each i. Further, let Y be the linear transformation such that $Yv_i = w_{i+1}$ for each $i < n$ and $Yv_n = Yw_j = 0$ for each j. We may represent the actions of X and Y on $V = V_n$ by the below diagram.

$$\begin{tikzcd}
v_1 \arrow[d, "X"] \arrow[dr, "Y"] & v_1 \arrow[d, "X"] \arrow[dr, "Y"] & \cdots \arrow[dr, "Y"] & v_n \arrow[d, "X"] \\
w_1 & w_2 & \cdots & w_n
\end{tikzcd}$$

So we see that $X^2 = Y^2 = XY = YX = 0$, but $(\mathrm{Id}_V + X)^p = (\mathrm{Id}_V + Y)^p = \mathrm{Id}_V$ as k has characteristic p. We thus make V into a kG-module via the representation $\varphi\colon x \mapsto \mathrm{Id}_V + X$, $y \mapsto \mathrm{Id}_V + Y$. Thus $\varphi(x)$ and $\varphi(y)$ respectively have the following matrices

$$\begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} \qquad \begin{pmatrix} I_n & 0 \\ N_n & I_n \end{pmatrix}$$

where I_n denotes the $n \times n$ identity matrix and $N = N_n$ is the $n \times n$ matrix whose only nonzero entries are 1s just below the diagonal. Then $\operatorname{End} V$ is simply the centraliser in $M_{2n}(k)$ of $\varphi(x)$ and $\varphi(y)$. It is then an exercise in linear algebra to check that any such matrix is of the form

$$\begin{pmatrix} A & 0 \\ C & A \end{pmatrix}$$

where $AN = NA$, with this latter condition being equivalent to A being a lower-triangular matrix with each diagonal constant (*i.e.* every entry on the diagonal is some fixed λ_1, every entry just below the diagonal is some fixed λ_2 and so on). Thus the subspace of $\operatorname{End} V$ of matrices with zero on the diagonal is a nilpotent ideal of $\operatorname{End} V$ of codimension one. As such, $\operatorname{End} V / \operatorname{rad} \operatorname{End} V$ has dimension one and in particular $\operatorname{End} V$ is local, thus by Theorem 2.32, V is indecomposable (note: this result was added in after the first lecture as it was needed here!). We have thus constructed an infinite family of indecomposable modules for G. Note that by Lemma 3.20 we have that any uniserial module is a quotient of some projective indecomposable kG-module and so in particular there are finitely many uniserial kG-modules. Thus without even looking closer at the structure of these modules we can tell that not every kG-module here is uniserial.

We now continue to build up machinery to allow us to look at the structure of projective modules. We now know that PIMs have simple heads and that any kG-module with irreducible head is in fact a quotient of the projective cover of that head. We would also like to do something similar by looking at submodules rather than quotients.

Recall from linear algebra that given a vector space V over k, the *dual space* V^* of V is the space $\mathrm{Hom}(V, k)$. If V is a kG-module, we may impose an action of G onto this space.

Definition 3.26
Let V be a kG-module. Then the *dual* of V is the kG-module $V^* := \mathrm{Hom}(V, k)$. This is the dual vector space of V with the action $g\varphi\colon v \mapsto \varphi(g^{-1}v)$.

Note that in the case of group representations, the dual V^* of a kG-module V is not the dual *as a kG-module* (this would be $\mathrm{Hom}_{kG}(V, kG)$), but the vector space dual with a G-action. All the typical correspondences between a vector space and its dual then carry over to the case of kG-modules, so $V^{**} \cong V$ and submodules of V correspond to quotients of V^* in the natural way. In this sense, taking the dual of a module is somehow 'turning it on its head.' Taking duals also distributes nicely over direct sums and vector space tensor products. In this vein, we have the following.

Lemma 3.27
V is irreducible if and only if V^* is irreducible.

Thus we also see that duality acts as a permutation (an involution, in fact) on $\mathrm{Irr}_k\, G$, and due to the friendly properties of duality we thus also see that the dual of a semisimple module remains semisimple. Now, since we still wish to look at projective modules, it is easiest to start with free ones. Since clearly $kG^* \cong kG$, we have

Lemma 3.28
Let V be a free module. Then V^* is free.

An immediate consequence of this is that the dual of a projective module is itself projective. Note for modules over arbitrary algebras this need not be the case.

Now, taking the dual of Theorem 2.38, we obtain the next proposition. In this case, duality typically means that we take any arrows that appear in any commutative diagrams and simply flip them around.

Proposition 3.29
Let U, V, I be kG-modules. Then the following are equivalent:

- *I is a direct summand of a free module.*
- *If $\varphi\colon I \hookrightarrow U$ then $\varphi(I)$ is a direct summand of U.*
- *If $\psi\colon U \hookrightarrow V$ and $\varphi\colon U \to I$ then there exists a homomorphism $\rho\colon V \to I$ such that $\varphi = \rho\psi$.*

As in the projective case, this last condition may be expressed by saying that the above diagram, with exact bottom row, commutes.

$$\begin{tikzcd} & I \\ V \arrow[ur, "\exists\rho"] & U \arrow[l, "\psi"] \arrow[u, "\varphi"'] & 0 \arrow[l] \end{tikzcd}$$

Modules I satisfying any of the above properties are called *injective*. When dealing with kG-modules, however, we have the following.

Theorem 3.30
A kG-module V is projective if and only if it is injective.

This above theorem is due to the fact that kG is a *quasi-Frobenius algebra* (or ring). A discussion of this may be found starting at [7, Definition 58.5], in particular the above theorem follows directly from [7, Theorem 58.14].

The following result now follows from Theorem 3.17 by duality since $(\operatorname{soc} P)^* \cong \operatorname{head} P^*$ is irreducible.

Corollary 3.31
If P is a projective indecomposable kG-module then $\operatorname{soc} P$ is irreducible.

Finally, we have this key result on the structure of the PIMs for kG-modules.

Theorem 3.32
Let P be a projective indecomposable kG-module. Then $\operatorname{head} P \cong \operatorname{soc} P$.

So with this, we now know that each PIM has irreducible head and socle, and that in fact these irreducible modules are isomorphic. So for a given $V \in \operatorname{Irr}_k G$, the projective cover $\mathcal{P}(V)$ looks like

$$\begin{matrix} V \\ \mathcal{H}(\mathcal{P}(V)) \\ V \end{matrix} \; .$$

In some cases, we can say more about the structure of the heart of projective indecomposable modules but in general they can be an incredibly complicated mess of extensions of many different irreducible modules that are practically impossible to untangle. In particular, no friendly picture of such a module is likely to exist. We shall however see some of the friendlier examples later in the course.

The proof of this theorem will not be included in lectures, but we shall go through it here for completeness as this is an important result and the proof isn't too long. We do, however, need to introduce a little bit of machinery specifically for this proof.

Let $(\; , \;)$ be the symmetric k-bilinear form on kG such that $(g, g^{-1}) = 1$ and $(g, h) = 0$ for all other choices of h. Here symmetric means that $(g, h) = (h, g)$ and bilinear means that for any $\lambda \in k$ we have that $(\lambda g, h) = (g, \lambda h) = \lambda(g, h)$. The form is also nondegenerate, meaning that there does not exist nonzero $x \in kG$ such that $(x, y) = 0$ for all $y \in kG$. We can also see that $(xy, z) = (x, yz)$ for all x, y, $z \in kG$.

Proof of Theorem 3.32. Let P be a projective indecomposable module. Then $\operatorname{head} P \cong V$ for some $V \in \operatorname{Irr}_k G$. Suppose that $\operatorname{soc} P \not\cong V$ and write $kG = Q \oplus R$ for $Q = P^{\oplus \dim V}$ so that R contains no summands isomorphic to P. Then $\operatorname{soc} Q \cong (\operatorname{soc} P)^{\oplus \dim V}$ and so in particular $\operatorname{Hom}_G(V, Q) = 0$. Then kG/R also has no submodules isomorphic to V. Let I be the sum of all submodules of kG isomorphic to V and note that $0 \neq I \leq R$. Now, the socle of $\mathcal{P}(V^*)^*$ must be isomorphic to V and $\mathcal{P}(V^*)^*$ must be a summand of R. Further, the submodule I is also an ideal of kG: as it is a submodule, it is a left ideal, and by Schur's Lemma (Lemma 2.9) $\varphi(I) \subseteq I$ for any $\varphi \in \operatorname{End} kG$ we have that I is a right ideal (since $\operatorname{End} kG \cong (kG)^o$). Let $J := \{\varphi \in \operatorname{End} kG \mid \operatorname{im} \varphi \subseteq I\}$, then $J \trianglelefteq \operatorname{End} kG$.

Let $\pi\colon kG \twoheadrightarrow Q$ be the natural projection (so $\ker \pi = R$ and take $\varphi \in J$. Now, $\varphi(kG) \leq I \leq R$ and so $\pi\varphi = 0$ and we claim that $\varphi\pi = \varphi$. As π is the identity on Q these maps clearly agree on Q and $\pi R = 0$ so $\varphi\pi$ vanishes on R. Suppose that $\varphi(R) \neq 0$. Then there exists $X \leq R$ such that $R/X \cong S$, and in fact there exists an indecomposable summand of R with this property. But such a summand must then be projective, contradicting our definition of R. So $\varphi(R) = 0$ and $\varphi = \varphi\pi - \pi\varphi = \varphi\pi$.

Now, we may suppose that $\varphi \neq 0$ and let $\alpha \in \operatorname{End} kG$. Thus $\varphi\alpha \in J$ and so $\varphi\alpha = \varphi\alpha\pi - \pi\varphi\alpha$. But endomorphisms of kG are simply right multiplications, so let a, b, $c \in kG$ such that $\alpha = \rho_a$, $\varphi = \rho_b$ and $\pi = \rho_c$ (where $\rho_x(y) := yx$) and we actually have that $ab = cab - abc$. But then

$$\begin{aligned}(a, b) &= (ab, 1)\\ &= (cab, 1) - (abc, 1)\\ &= (c, ab) - (ab, c)\\ &= 0\end{aligned}$$

but this must hold for all $a \in kG$. Since our form is nondegenerate, this forms $b = 0$ and thus $\varphi = 0$. This contradiction completes the proof. ■

Another common tool from linear algebra which we shall be co-opting here is that of tensor products. Much like the dual space, we may equip the tensor product $V \otimes_k W$ with a G-action which happens to work nicely with all of our constructions.

Definition 3.33
Let V and W be kG-modules. Then the *tensor product* $V \otimes W$ of V and W is the kG-module with the underlying vector space $V \otimes_k W$ under the G-action $g(v \otimes w) = gv \otimes gw$. We also analogously define the n-fold tensor product $V^{\otimes n} := \bigotimes_{i=1}^n V$.

It is very important once again to note that the tensor product of kG-modules is in fact just a tensor product taken over k with a G-action, and *not* the module $V \otimes_{kG} W$. This tensor product construction is going to come in particularly useful when we wish to describe the projective modules for $\mathrm{SL}_2(p)$.

Recall that a kG-module is faithful if the only element of G which fixes the whole space is the identity. Also, for kG-modules V and W we use the notation $V \mid W$ to mean there exists some submodule U of W such that $W \cong U \oplus V$, *i.e.* V is isomorphic to a direct summand of W.

Theorem 3.34
Let V be a faithful kG-module and P a projective indecomposable kG-module. Then $P \mid V^{\otimes n}$ for some $n \geq 1$.

This means that given a faithful kG-module, we can actually find every projective indecomposable kG-module as summands of its tensor powers. This can be a very powerful result provided it is actually feasible to deconstruct these tensor products, which can in general be very difficult. This next result also tells us that it is typically sufficient to only look at tensors of simple modules.

Proposition 3.35
If G has no nontrivial normal p-subgroups then $\bigoplus_{V \in \operatorname{Irr}_k G} V$ is a faithful kG-module.

Note that if G does have a nontrivial normal p-subgroup Q then this subgroup lies in the kernel of every irreducible kG-module and thus also lies in the kernel of the above direct sum. For example, the groups $\mathrm{SL}_2(q)$ for q an odd prime power all have a centre of order 2 and so, in particular, if k has characteristic 2 then $\mathrm{SL}_2(q)$ has no faithful irreducible kG-modules.

This next result is an example of *Tensor-hom adjunction*, a property that one might see if studying homological algebra. In that setting, this is a statement about the *functors* $-\otimes V$ and $\mathrm{Hom}(V, -)$. We will definitely not be looking any further into that here, however.

Lemma 3.36
Let U, V and W be kG-modules. Then $\mathrm{Hom}_G(U \otimes V, W) \cong \mathrm{Hom}_G(U, V^* \otimes W)$.

Finally, we have this useful result regarding projective modules. To prove the following result, it is sufficient to show that $V \otimes kG$ is free for any kG-module V. One can also prove this result much easier using some future results, but we wish to make use of this lemma here.

Lemma 3.37
Let V be a kG-module and P a projective kG-module. Then $V \otimes P$ is projective.

We now conclude this section by returning to $\mathrm{SL}_2(p)$ and determining the structure of its projective indecomposable modules. Recall the construction of the irreducible $k\,\mathrm{SL}_2(p)$-modules from Example 3.13.

Example 3.38
Set $G := \mathrm{SL}_2(p)$ and let $V_1, \ldots, V_p$ denote the irreducible kG-modules as before. We shall see later that if a kG-module V is projective upon restriction to a subgroup of G containing a Sylow p-subgroup, then V is projective as a kG-module. It is, however, far too early for that, so you shall simply have to believe it for now. We shall use this result to show, first, that V_p is projective.

Let $Q \in \mathrm{Syl}_p G$. Then Q is cyclic of order p and we may suppose that Q consists of the upper unitriangular matrices in G. It was already demonstrated in Example 3.13 that $(V_p)_Q$ is uniserial, and in fact since $\dim V_p = p$ we have that V_p is a projective indecomposable module for Q (indeed, it is kQ itself).

Now, let $P_i := \mathcal{P}(V_i)$. We have shown that $P_p \cong V_p$. To go further, we will investigate particular tensor products of irreducible modules. In particular, we show the following.

Lemma 3.39
Suppose that $2 \leq n < p$. Then $V_2 \otimes V_n \cong V_{n-1} \oplus V_{n+1}$.

Proof. Recall that V_n is the space of homogeneous polynomials in X and Y of degree $n-1$. As such, we have an obvious surjective homomorphism $\psi\colon V_2 \otimes V_n \to V_{n+1}$ given by $f \otimes g \mapsto fg$.

We first prove that the kernel of this map is isomorphic to V_{n-1} and, checking dimensions, it is sufficient to provide an embedding of V_{n-1} into this kernel. Define a linear map $\varphi\colon V_{n-1} \to V_2 \otimes V_n$ by $\varphi(f) := X \otimes Yf - Y \otimes Xf$, so it is plain to see that $\varphi\colon V_{n-1} \to \ker\psi$. Now, if s is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ then of course $ad - bc = 1$ and we have

$$\begin{aligned}
s(\varphi(f)) &= s(X \otimes Yf - Y \otimes Xf) \\
&= s(X) \otimes s(Y)s(f) - s(Y) \otimes s(X)s(f) \\
&= (aX + cY) \otimes (bX + dY)s(f) - (bX + dY) \otimes (aX + cY)s(f) \\
&= (ad - bc)(X \otimes Ys(f)) - (ad - bc)(Y \otimes Xs(f)) \\
&= \varphi(s(f))
\end{aligned}$$

so φ is a homomorphism. The elements X^iY^{n-2-i} for $0 \leq i \leq n-2$ form a basis for V_{n-1} and

$$\varphi(X^iY^{n-2-i}) = X \otimes X^iY^{n-1-i} - Y \otimes X^{i+1}Y^{n-2-i}.$$

The $2(n-1)$ tensors whose differences yield the $n-1$ images above (*e.g.* in the above case the tensors in question would be $X \otimes X^iY^{n-1-i}$ and $Y \otimes X^{i+1}Y^{n-2-i}$) then form subsets of bases for $X \otimes V_n$ and $Y \otimes V_n$, respectively, and thus are linearly independent. In particular, φ is injective. Thus $\ker\psi \cong V_{n-1}$

It now remains only to show that $V_2 \otimes V_n$ has a submodule isomorphic to V_{n+1}. We prove this by induction, but backwards. Let $n = p-1$. Then $V_2 \otimes V_n$ has the projective module V_p as a homomorphic image and thus as a direct summand. Now let $n+1 < p$ and suppose that $V_2 \otimes V_{n+1}$ is as claimed in the lemma. Since V_{n+1} is irreducible, it is sufficient to show that $\mathrm{Hom}_G(V_{n+1}, V_2 \otimes V_n) \neq 0$ as any nonzero homomorphism will be an embedding. Now, V_2^* is a 2-dimensional irreducible kG-module and thus isomorphic to V_2 itself. Further, by Lemma 3.36 we have

$$\begin{aligned}
\mathrm{Hom}_G(V_{n+1}, V_2 \otimes V_n) &\cong \mathrm{Hom}_G(V_{n+1} \otimes V_2^*, V_n) \\
&\cong \mathrm{Hom}_G(V_{n+1} \otimes V_2, V_n) \\
&\cong \mathrm{Hom}_G(V_n \oplus V_{n+2}, V_n)
\end{aligned}$$

by induction, and this last Hom is clearly nonzero. ∎

Now, the module $V_2 \otimes V_p$ is projective by Lemma 3.37 and we may replicate the first part of the proof of the above lemma. Thus $V_2 \otimes V_p$ has a submodule isomorphic to V_{p-1} and the quotient by this submodule is isomorphic to V_{p+1} (note this module was defined in Example 3.13 but is reducible). Thus $V_2 \otimes V_p$ has a summand isomorphic to P_{p-1}. As $\dim V_{p-1} = p-1$ is not divisible by p, V_{p-1} is not projective by Corollary 3.22. As V_{p-1} is the unique simple submodule of P_{p-1} we have a chain $V_{p-1} = \mathrm{soc}\, P_{p-1} \leq \mathrm{rad}\, P_{p-1} \leq P_{p-1}$ and so $\dim P_{p-1} \geq 2(p-1)$. Since $p \mid \dim P_{p-1}$ again by Corollary 3.22 and $\dim(V_2 \otimes V_p) = 2p$ we thus have that $V_2 \otimes V_p \cong P_{p-1}$ provided $p > 2$. (Note if $p = 2$, $P_{p-1} = P_1$ and $\dim kG = 6 = \dim P_1 + 2\dim P_2$ so $\dim P_1 = 2$ and thus P_1 is the uniserial module given by the unique non-split extension $0 \to k \to P_1 \to k \to 0$.) Returning to the case $p > 2$, we have that $V_2 \otimes V_p \cong P_{p-1}$ and so the socle and head are both isomorphic to V_{p-1} and we have that $\dim \mathcal{H}(P_{p-1}) = 2$. We claim that $\mathcal{H}(P_{p-1}) \cong V_2$. Since $V_2 \otimes V_p / V_{p-1} \cong V_{p+1}$ this is equivalent to showing that V_{p+1} has a submodule isomorphic to V_2. The map $\sigma\colon k[X,Y] \to k[X,Y]$ with $\sigma(X) = X^p$ and $\sigma(Y) = Y^p$ is an injective endomorphism of $k[X,Y]$ which maps V_2 into V_{p+1}, thus we are done.

Next, we look at P_{p-2} by studying $M \coloneqq V_2 \otimes P_{p-1}$. As one of the tensor factors is projective, this module is projective and using the structure of P_{p-1} described above we find a series of submodules with successive quotients isomorphic to $V_2 \otimes V_{p-1}$, $V_2 \otimes V_2$ and $V_2 \otimes V_{p-1}$ or, using Lemma 3.39, $V_{p-2} \oplus V_p$, $V_1 \oplus V_3$, $V_{p-2} \oplus V_p$. Since V_p is projective and V_{p-2} lies in head M, we have that M has a summand isomorphic to $P_{p-2} \oplus V_p^{\oplus 2}$ (and, if $p = 3$, $P_1 \oplus V_3^{\oplus 3}$). As before, P_{p-2} has two composition factors isomorphic to V_{p-2} and has dimension divisible by p and its only other possible composition factors are V_1 or V_3 (just V_1 when $p = 3$). We therefore require that $\mathcal{H}(P_{p-2})$ has composition factors V_1 and V_3 (again, if $p = 3$ there is only V_1, we will investigate this later but the structure will be as expected). We claim that in fact $\mathcal{H}(P_{p-2}) \cong V_1 \oplus V_3$. Were the heart not semisimple it would need to be uniserial and in particular would not be self-dual. However, since V_{p-2} is self-dual, so is P_{p-2} and it follows that $\mathcal{H}(P_{p-2})$ is thus also self-dual.

We determine the structure of P_n for $2 < n < p-1$ by induction (but backwards, again, taking the above as our base case). Suppose that $V_2 \otimes P_n$ has a series of submodules with successive quotients $V_2 \otimes V_n$, $V_2 \otimes (V_{p+1-n} \oplus V_{p-1-n})$ and $V_2 \otimes V_n$, reducing via Lemma 3.39 to $V_{n-1} \oplus V_{n+1}$, $V_{p+2-n} \oplus V_{p-n} \oplus V_{p-n} \oplus V_{p-n-2}$ and $V_{n-1} \oplus V_{n+1}$ (where if the subscript is zero we delete the module). Since $V_2 \otimes P_n$ is projective with a quotient isomorphic to $V_{n+1} \oplus V_{n-1}$ we have that $V_2 \otimes P_n \cong P_{n+1} \oplus U$ where U is projective with quotient V_{n-1} and has composition factors V_{n-1}, V_{p+2-n}, V_{p-n} and V_{n-1}. Proceeding as in the above paragraph we see that $U \cong P_{n-1}$ and its structure is again as claimed.

Finally, we deal with P_1 by looking at $V_2 \otimes P_2$. Again we obtain $V_2 \otimes P_2 \cong P_3 \otimes V_p \oplus U$ for some projective kG-module U with head V_1 and composition factors V_1, V_{p-2}, V_1. Clearly in this case $U \cong P_1$ is uniserial with these composition factors. We have thus shown that the projective indecomposable $\mathrm{SL}_2(p)$-modules are as below for $1 < n < p-1$. Note that when $p = 2$ we have $p - 2 = 0$ and thus we simply delete V_{p-2} from the picture, yielding P_1 of dimension two.

$$P_1 \sim \begin{matrix} V_1 \\ V_{p-2} \\ V_1 \end{matrix}, \qquad P_n \sim \begin{matrix} V_n \\ V_{p-1-n} \oplus V_{p+1-n} \\ V_n \end{matrix}, \qquad P_{p-1} \sim \begin{matrix} V_{p-1} \\ V_2 \\ V_{p-1} \end{matrix}, \qquad P_p \cong V_p.$$

4 Modular representations

In this section, we begin looking at certain structures which are typically trivial or otherwise not noteworthy in the case where $p \nmid |G|$. We will begin by introducing induced modules (which are also very useful in coprime characteristics) and use these along with other structures to allow us to use information about p-local subgroups (normalisers of p-subgroups) of G to tell us about the group's representation theory. This general idea is fundamental to modular representation theory, and various questions relating p-subgroups to the representation theory of G are some of the largest open problems in this area.

We saw before a characterisation of free modules using a linear subspace of the module as a *basis*. A linear subspace of a kG-module can also be regarded as a $k1$-subspace for the subgroup $1 \leq G$. We can generalise this notion of free to depend on subgroups of G.

Definition 4.1
For a subgroup $H \leq G$, we say that a kG-module V is *relatively H-free* if there exists a kH-

submodule $X \leq V$ such that any kH-homomorphism of X to any kG-module W extends uniquely to a kG-homomorphism of V to W.

Compare the above definition with Proposition 2.37 and see that if we set $H = 1$ then we recover the usual definition of freeness. We say that V is relatively H-free with respect to X (and we may abuse this notation by using X to refer to an isomorphism class of kH-modules rather than a specific fixed submodule).

Lemma 4.2
Let V, W be relatively H-free kG-modules with respect to the kH-submodules X and Y, respectively, with $X \cong Y$. Then $V \cong W$.

Proposition 4.3
Suppose that X is a kH-module for $H \leq G$. Then there exists a kG-module which is relatively H-free with respect to X.

The module satisfying the above proposition that any proof typically constructs is called the *induced module* of X from H to G. A typical proof of the above proposition is given on [1, p. 55] in which the induced module is constructed. We will be using a different definition for these modules, coming after the next corollary.

Corollary 4.4
Let V be a kG-module generated by the kH-submodule X for $H \leq G$. Then V is relatively H-free with respect to X is and only if $\dim V = [G : H] \dim X$.

Definition 4.5
Let $H \leq G$ and let U be a kH-module. Then the *induced module* of U from H to G is the kG-module $\operatorname{Ind}_H^G U := kG \otimes_{kH} U$.

Note that the tensor product above is specifically the tensor over kH, not the tensor over k as we have seen before. One way of regarding this is to take the typical vector space tensor product $kG \otimes U$ and quotient out by the subspace spanned by elements of the form $ah \otimes v - a \otimes hv$ for $a \in kG$, $h \in H$ and $v \in U$. We equip this space with the structure of a kG-module by setting $g(a \otimes v) := (ga) \otimes v$ (note this is not the usual action on the space $kg \otimes U$). In Alperin's book, the induced module is denoted U^G, but that would clash with our definition of a fixed space so we use the clearer (but unfortunately more clunky) notation seen above.

Lemma 4.6
Let $H \leq G$ and V be a kH-module. Then $\operatorname{Ind}_H^G V$ is relatively H-free with respect to V and is also equal (as vector spaces) to the direct sum

$$\operatorname{Ind}_H^G V = \sum_{s \in G/H} s \otimes V$$

where $\dim(s \otimes V) = \dim V$.

We now provide a selection of properties of induced modules, the proofs for all of these may be found in [1, §8] but are mostly not provided here.

Lemma 4.7
Let V, V_1 and V_2 be kH-modules for $H \leq G$ and let W be a kG-module. Then

i) *If V is free (resp. projective) then $\operatorname{Ind}_H^G V$ is free (resp. projective).*

ii) $\operatorname{Ind}_H^G(V_1 \oplus V_2) \cong \operatorname{Ind}_H^G V_1 \oplus \operatorname{Ind}_H^G V_2$.

iii) $\operatorname{Ind}_H^G(V^*) \cong (\operatorname{Ind}_H^G V)^*$.

iv) *If U is a kL-module for $L \leq H$ then $\operatorname{Ind}_H^G(\operatorname{Ind}_L^H U) \cong \operatorname{Ind}_L^G U$.*

v) $W \otimes \operatorname{Ind}_H^G V \cong \operatorname{Ind}_H^G(W_H \otimes V)$.

The first two parts of the next lemma are typically called *Frobenius Reciprocity* when $p \nmid |G|$ and *Frobenius–Nakayama Reciprocity* otherwise and is an incredibly useful result.

Lemma 4.8
Recall the notation from above. Then

i) $\operatorname{Hom}_G(\operatorname{Ind}_H^G V, W) \cong \operatorname{Hom}_H(V, W_H)$.

ii) $\operatorname{Hom}_G(W, \operatorname{Ind}_H^G V) \cong \operatorname{Hom}_H(W_H, V)$.

iii) *If $\gamma \in \operatorname{Hom}_H(W_H, V)$ then the map sending $w \in W$ to $\sum_{s \in G/H} s \otimes \gamma(s^{-1}w)$ lies in $\operatorname{Hom}_G(U, \operatorname{Ind}_H^G V)$ and this yields an isomorphism of $\operatorname{Hom}_H(W_H, V)$ onto $\operatorname{Hom}_G(W, \operatorname{Ind}_H^G V)$.*

iv) *If $\alpha \in \operatorname{Hom}_H(V_1, V_2)$ then there exists a unique homomorphism, denoted by $\operatorname{Ind}_H^G \alpha$, in $\operatorname{Hom}_G(\operatorname{Ind}_H^G V_1, \operatorname{Ind}_H^G V_2)$ extending α.*

v) *If the sequence*

$$0 \to V_1 \xrightarrow{\alpha} V \xrightarrow{\beta} V_2 \to 0$$

is exact then so is

$$0 \to \operatorname{Ind}_H^G V_1 \xrightarrow{\operatorname{Ind}_H^G \alpha} \operatorname{Ind}_H^G V \xrightarrow{\operatorname{Ind}_H^G \beta} \operatorname{Ind}_H^G V_2 \to 0$$

and one splits if and only if the other does.

Recall that we say a short exact sequence is *split* if the middle term is isomorphic to the direct sum of the other two terms, and the maps correspond to the natural injection and projection maps, respectively.

Proof. We prove only the first two statements. The first statement follows from the definition of relative freeness: Each $\alpha \in \operatorname{Hom}_H(V, W_H)$ extends uniquely to $\hat{\alpha} \in \operatorname{Hom}_G(\operatorname{Ind}_H^G V, W)$ and such a map is surjective since restricting a map and re-extending it yields the same homomorphism. To obtain the second statement:

$$\begin{aligned}
\operatorname{Hom}_G(W, \operatorname{Ind}_H^G V) &\cong \operatorname{Hom}_G((\operatorname{Ind}_H^G V)^*, W^*) \\
&\cong \operatorname{Hom}_G(\operatorname{Ind}_H^G(V^*), W^*) \\
&\cong \operatorname{Hom}_H(V^*, (W^*)_H) \\
&\cong \operatorname{Hom}_H(W_H, V)
\end{aligned}$$

using part i), the previous lemma and the fact that $\operatorname{Hom}(W, V) \cong \operatorname{Hom}(V^*, W^*)$. ■

Next up we have Mackey's Theorem. We'll call it a lemma here, but it is a very useful result if we happen to know well the structure of the modules for a proper subgroup of G and the action of G on our subgroup H. First, though, if V is a kH-module then $s \otimes V$ is a summand of $\operatorname{Ind}_H^G V$ and may be regarded as a module for $H^{s^{-1}}$ via the action $shs^{-1}(s \otimes v) = (sh) \otimes v = s \otimes hv$ defined by the action of H on V. In general, we let $s(V)$ denote a $k(H^{s^{-1}})$-module obtained as $s \otimes V$ for some kH-module V. This phenomenon is occasionally referred to as 'transport of structure.'

Lemma 4.9 (Mackey's Theorem)
Let L, $H \leq G$. Then

$$(\operatorname{Ind}_H^G V)_L \cong \bigoplus_{s \in L\backslash G/H} \operatorname{Ind}_{L \cap H^{s^{-1}}}^L s(V).$$

A notable special case of this lemma is of course when $L = H$, then we get

$$(\operatorname{Ind}_H^G V)_L \cong \bigoplus_{s \in L\backslash G/H} \operatorname{Ind}_{H \cap H^{s^{-1}}}^H s(V)$$

which can sometimes be easy to work out if we know a lot about the intersections of H with its conjugates.

The proof of Mackey's Theorem is essentially just careful bookkeeping as one moves between cosets and double cosets. We include it here for completeness.

Proof. Let T_s be a set of left coset representatives for $L \cap H^{s^{-1}}$ in L. Then $LsH = \bigsqcup_{t \in T_s} tsH$, so that, as vector spaces,

$$\operatorname{Ind}_H^G V = \bigoplus_{\substack{s \in L\backslash G/H \\ t \in T_s}} ts \otimes V.$$

Since each t lies in L, $\bigoplus_{t \in T_s} ts \otimes V$ is a kL-module for each s. Denote this by V_s. Then $(\operatorname{Ind}_H^G V)_L = \bigoplus_{s \in L\backslash G/H} V_s$ so it suffices to prove that $V_s \cong \operatorname{Ind}_{L \cap H^{s^{-1}}}^L s(V)$. Since $|T_s| = [L : L \cap H^{s^{-1}}]$ we have that $\dim V_s = [L : L \cap H^{s^{-1}}] \dim V$ and further $s \otimes V \subseteq V_s$ with $s \otimes V$ isomorphic to $s(V)$ as $k(L \cap H^{s^{-1}})$-modules. We thus actually only need to show that V_s is generated by $s \otimes V$ as a kL-module by Corollary 4.4. But $LsH = T_s sH$, so this is clear. ■

Next up we have Green's indecomposability criterion. The proof of this theorem is too long to include here in full, but is [1, §8, Theorem 8].

Theorem 4.10
Let $N \trianglelefteq G$ with G/N a p-group. If V is an indecomposable kN-module then $\operatorname{Ind}_N^G V$ is indecomposable.

The key observation in the proof is that since G/N is a p-group, there exists a chain of subgroups $N = G_r \lneq \cdots \lneq G_1 \lneq G_0 = G$ where $[G_i : G_{i+1}] = p$. By induction, one may assume that $[G : N] = p$ and this is then the meat of the proof.

Now that we have some conditions for induced modules, we take a quick look at some. We promised before that we would generalise the concept of permutation modules, and this is that.

Example 4.11
Let V be the permutation module of G on a subgroup $H \leq G$. Then by Lemma 4.6 we have that $V \cong \operatorname{Ind}_H^G k$. We can also observe here the previous claim that V^G is a 1-dimensional space spanned by the sum of the standard basis elements. Since k is a 1-dimensional module, $\dim \operatorname{Hom}_G(k, V) = \dim V^G$. Now, using Frobenius–Nakayama Reciprocity (Lemma 4.8) we see that $\dim \operatorname{Hom}_G(k, V) = \dim \operatorname{Hom}_G(k, \operatorname{Ind}_H^G k) = \dim \operatorname{Hom}_G(k_H, k) = 1$ so that $\dim V^G = 1$ and the fixed space we demonstrated earlier is the only one.

It is also common to use induced modules from various subgroups to find the irreducible modules for more complicated groups. For example, many of the irreducible modules for the groups of Lie type in cross characteristic (think $G = \mathrm{PSL}_n(q)$ where $p \nmid q$) may be obtained via *Harish–Chandra induction* which involves taking modules for well-known lower-rank subgroups, extending them to a slightly larger group and then inducing these modules up to the whole group. Many of the irreducible modules for these groups may then be located as composition factors of such modules.

A similar approach can also be used to yield the irreducible modules for the symmetric groups (at least in coprime characteristic), where one induces modules from certain *Young subgroups* to find new irreducible modules as particular submodules (called *Specht modules*) of these induced modules. When $p \mid |S_n|$ this is much more complicated, as the modules found via this method need not be irreducible.

Now that we have generalised freeness and investigated induced modules, we shall also generalise projectivity. This will allow us to make an important connection between kG-modules and p-subgroups of G. As we have previously done, we shall define this new property via a proposition full of equivalent conditions.

Proposition 4.12
Let V be a kG-module and $H \leq G$. Then the following are equivalent:

- *V is a direct summand of a relatively H-free module.*
- *If $\varphi\colon U \to V$ is split as a kH-homomorphism then φ is split as a kG-homomorphism.*
- *If $\varphi\colon U \to W$ and $\psi\colon V \to W$ then there is a kG-homomorphism $\rho\colon V \to U$ with $\varphi\rho = \psi$ provided there is a kH-homomorphism with this property.*
- *$V \mid \operatorname{Ind}_H^G(V_H)$.*

Any kG-module V satisfying any (thus all) of the above equivalent properties is said to be *relatively H-projective* and all of the other typical abuses of notation and rearrangings that come with it. We now plan to investigate for which subgroups of G is a given kG-module relatively H-projective. First, we prove a theorem promised earlier during our example in which we worked out the projective $k\,\mathrm{SL}_2(p)$-modules (Example 3.38).

Theorem 4.13
Suppose $H \leq G$ contains a Sylow p-subgroup of G. Then every kG-module V is relatively H-projective.

Proof. If $p \nmid |G|$ then the trivial subgroup is a Sylow p-subgroup and by previous discussions we have that every kG-module is relatively H-projective for all $H \leq G$. Thus suppose that $p \mid |G|$ so that a Sylow p-subgroup of G is nontrivial and that H contains such a subgroup.

Suppose that $\varphi\colon V \to U$ is a homomorphism of kG-modules which splits as a kH-module homomorphism. We must show that φ splits as a kG-module homomorphism, then we will be done by Proposition 4.12. Let $W := \ker\varphi$. Then as kH-modules we have $V = W \oplus U$. Let $\pi\colon V \twoheadrightarrow W$ be a projection map onto W which is a kH-homomorphism. Now, as we saw in the proof of Maschke's Theorem (Theorem 3.7), set

$$\pi' := \frac{1}{[G:H]} \sum_{s\in G/H} \pi^s$$

note that since H contains a Sylow p-subgroup of G we have that $[G:H]$ is invertible in k. Proceeding exactly as in the proof of Maschke's Theorem one verifies that the above map is a homomorphism of kG-modules and so we are done. ■

In particular, the corollary to the above theorem that we used when working out the projective $k\,\mathrm{SL}_2(p)$-modules is the following.

Corollary 4.14
Suppose $H \leq G$ contains a Sylow p-subgroup of G and V is a kG-module such that V_H is projective. Then V is projective.

At this point, we are going to be talking about direct summands frequently, so we recall the notation from before that for kG-modules U and V we say that $U \mid V$ if U is a direct summand of V, *i.e.* there exists $W \leq V$ and $\varphi\colon U \hookrightarrow V$ such that $V = \varphi(U) \oplus W$.

Theorem 4.15
Let V be an indecomposable kG-module. Then there is a p-subgroup Q of G, unique up to G-conjugacy, such that V is relatively H-projective for $H \leq G$ if and only if H contains a conjugate of Q.

Moreover, there is an indecomposable kQ-module S, unique up to $N_G(Q)$-conjugacy, such that $V \mid \mathrm{Ind}_Q^G S$.

Definition 4.16
In the above theorem, we call the subgroup Q given above a *vertex* of V and the indecomposable module S a *source* of V.

The goal of this definition is that we may use vertices to attempt to 'measure' the distance from projectivity — the smaller Q is, the 'closer' our module V is to being projective.

Proof of Theorem 4.15. Since V is relatively projective for a Sylow p-subgroup of G, there exists a p-subgroup Q of minimal order such that V is relatively Q-projective. In particular, since V is indecomposable this means that $V \mid \mathrm{Ind}_Q^G(V)$ and there exists some $S \mid V_Q$ such that $V \mid \mathrm{Ind}_Q^G S$. If $Q \leq H \leq G$ then $V \mid \mathrm{Ind}_H^G(\mathrm{Ind}_Q^H S)$ and so V is relatively H-projective. This also tells us that V is relatively H^g-projective for any $g \in G$ since $V \cong g^{-1}(V)$.

Now suppose that V is relatively H-projective for $H \leq G$ and U is an indecomposable kH-module such that $V \mid \mathrm{Ind}_H^G U$. Since $V \mid \mathrm{Ind}_Q^G S$, clearly $V \mid \mathrm{Ind}_Q^G(g(S))$ for any $g \in G$. Moreover, if $g \in N_G(Q)$ then $g(S)$ is a kQ-module.

Now, $S \mid V_Q$ and $V \mid \operatorname{Ind}_H^G U$ so $S \mid (\operatorname{Ind}_H^G U)_Q$ and Mackey's Theorem (Lemma 4.9) yields

$$(\operatorname{Ind}_H^G U)_Q \cong \bigoplus_{s \in Q\backslash G/H} \operatorname{Ind}_{Q\cap H^s}^Q (s^{-1}(U)).$$

Hence $S \mid \operatorname{Ind}_{Q\cap H^s}^Q (s^{-1}(U))$ for some $s \in G$. But then $V \mid \operatorname{Ind}_{Q\cap H^s}^G$ since $V \mid \operatorname{Ind}_Q^G S$ and so the minimality of the order of Q forces $Q = Q \cap H^s$ so that $Q^{s^{-1}} \subseteq H$ as required. Further, if $Q = H$ then $Q \cap Q^s = Q$ means that $s \in N_G(Q)$ so that $U = s(S)$ as claimed. ■

Example 4.17
Now, suppose that P is a Sylow p-subgroup of G and let Q be a vertex for the trivial module. Since k_Q is of course also 1-dimensional and trivial, it is indecomposable and thus a source for k. Thus

$$k_P \mid (\operatorname{Ind}_Q^G k)_P \cong \bigoplus_{s \in P\backslash G/Q} \operatorname{Ind}_{P\cap Q^s}^P (s(k_Q))$$

so $k_P \mid \operatorname{Ind}_{P\cap Q^s}^P$ for some $s \in G$. Set $R := P \cap Q^s$. Now, $\operatorname{soc} \operatorname{Ind}_R^P k = (\operatorname{Ind}_R^P k)^P$ as P is a p-group, and this fixed space has dimension $\dim \operatorname{Hom}_P(k, \operatorname{Ind}_R^P k) = \dim \operatorname{Hom}_R(k,k) = 1$. As $\operatorname{soc} \operatorname{Ind}_R^P k$ is irreducible, the module itself must be indecomposable. Thus we have that $R = P$ so that $P = Q^s$ and so Q is a Sylow p-subgroup of G. As a corollary of this, we learn that all of the Sylow p-subgroups of G are conjugate, just in case we didn't already know.

So what we have shown above is that given an arbitrary group G, the vertex for the trivial module is a Sylow p-subgroup of G. By our previous description, this means that the trivial module is always as far from being projective as possible.

We now provide a few more properties of vertices and sources.

Lemma 4.18
Suppose V is an indecomposable kG-module with vertex Q and $Q \leq H \leq G$. Then there is an indecomposable kH-module U which satisfies any two of the following:

i) $U \mid V_H$.

ii) $V \mid \operatorname{Ind}_H^G U$.

iii) U has vertex Q.

One can actually find a module that satisfies all three of the above, but the proof of that statement requires the Green correspondence, which in turn requires the lemma as stated above.

Proof. Since V is relatively H-projective, $V \mid \operatorname{Ind}_H^G V$ and so in particular there exists indecomposable $U \mid V_H$ such that $V \mid \operatorname{Ind}_H^G U$. This module U satisfies the first two points.

Now suppose that S is a kQ-module which is a source for V so that $V \mid \operatorname{Ind}_Q^G S$. Now, $\operatorname{Ind}_Q^G S = \operatorname{Ind}_H^G(\operatorname{Ind}_Q^H S))$ and so there exists indecomposable $U \mid \operatorname{Ind}_Q^H S$ with $V \mid \operatorname{Ind}_H^G U$. We claim that U has vertex Q so that U satisfies the final two points. Since $U \mid \operatorname{Ind}_Q^H S$ we have that U is relatively Q-projective and thus a vertex R of U lies in Q. Let W be a kR-module with $U \mid \operatorname{Ind}_R^H W$. Then

$\operatorname{Ind}_H^G V \mid \operatorname{Ind}_H^G(\operatorname{Ind}_R^H W)) = \operatorname{Ind}_R^G W$ so that $V \mid \operatorname{Ind}_R^G W$ and V is also relatively R-projective. Thus R contains a conjugate of Q, yet $R \leq Q$ and so they are equal.

Finally, let S be an indecomposable kQ-module with $S \mid V_Q$ and $V \mid \operatorname{Ind}_Q^G S$. Then there is an indecomposable kH-module U with $U \mid V_H$ and $S \mid U_Q$. If U has vertex Q then it satisfies conditions i) and iii).

To see this, note that $U \mid V_H$ so $U \mid (\operatorname{Ind}_Q^G S)_H$ and by Mackey's theorem (Lemma 4.9) there exists $s \in G$ such that $U \mid \operatorname{Ind}_{H\cap Q^s}^H(s^{-1}(S))$. Then U has vertex R with $R \subseteq H \cap Q^s$ and we need only show that R is H-conjugate to Q. Now, U is a summand of a module induced from R to H and $S \mid U_Q$ so that S is relatively $Q \cap R^h$-projective for some $h \in H$, but S has vertex Q so $Q \cap R^h$ cannot be proper in Q so that $Q \leq R^h$. But $R \leq Q^s$ and so $Q = R^h$ as required. ∎

Lemma 4.19
Let V be a relatively Q-projective kH-module for $Q \leq H \leq G$. Then $(\operatorname{Ind}_H^G V)_H \cong V \oplus W$ where every indecomposable summand of W is relatively projective for a subgroup of the form $Q^s \cap H$ for $s \in G \setminus H$.

Proof. Since V is relatively Q-projective, there exists a kQ-module U such that $V \mid \operatorname{Ind}_Q^H U$ and so $\operatorname{Ind}_Q^H U \cong V \oplus T$ for some kH-module T. So $\operatorname{Ind}_Q^G U \cong \operatorname{Ind}_H^G V \oplus \operatorname{Ind}_H^G T$ and by Mackey's theorem (Lemma 4.9) we have

$$(\operatorname{Ind}_Q^G U)_H \cong \bigoplus_{s \in H\backslash G/Q} \operatorname{Ind}_{H\cap Q^s}^H s^{-1}(U) \cong \operatorname{Ind}_Q^H U \oplus Y$$

where Y is the direct sum of all the terms above corresponding to $s \notin H$, meaning that each summand of Y is relatively projective for a subgroup of the form $H \cap Q^s$ as claimed. ∎

Lemma 4.20
Suppose that V is an indecomposable kG-module with vertex Q and trivial source. If $H \leq G$ then V_H has an indecomposable summand with a vertex containing $Q \cap H$.

Proof. Since V has vertex Q and source k we have that $k \mid V_Q$ so that $k \mid V_{Q\cap H}$. As such, there is an indecomposable summand U of V_H with $k \mid U_{Q\cap H}$. Let R be a vertex of U. Since U is relatively R-projective, Mackey's theorem (Lemma 4.9) tells us that every indecomposable summand of $U_{Q\cap H}$ is relatively projective for some subgroup of the form $(Q \cap H) \cap R^h$ for $h \in H$. But $k \mid U_{Q\cap H}$ and has vertex $Q \cap H$, so an H-conjugate of $Q \cap H$ lies in R^h and so we are done. ∎

Lemma 4.21
Let V be a kG-module with V_N indecomposable for $N \trianglelefteq G$. If Q is a vertex of V then QN/N is a Sylow p-subgroup of G/N.

Proof. Let $S \in \operatorname{Syl}_p(G)$ such that $Q \leq S$ and $SN/N \in \operatorname{Syl}_p G/N$ contains QN/N. By Lemma 4.18, there is an indecomposable summand of V_{SN} with vertex Q. However, V_N is indecomposable and thus so is V_{SN}. In particular, V_{SN} is relatively QN-projective and $V \mid \operatorname{Ind}_{QN}^{SN} V$. Since $QN/N \leq SN/N$ are p-groups, there is a series of subgroups connecting them such that each term in the series is normal in the preceding one. Hence, iterating Green's Indecomposability Criterion (Theorem 4.10) we have that $\operatorname{Ind}_{QN}^{SN} V$ is indecomposable so $V_{SN} \cong \operatorname{Ind}_{QN}^{SN} V$. Since $\dim(\operatorname{Ind}_{QN}^{SN} V) = [SN : QN] \dim V$ we have that $SN = QN$ as required. ∎

We now plan to look at a special case of the Green correspondence which is vastly more accessible than the full force of the actual result yet still applicable to our main $\mathrm{SL}_2(p)$ example. For this, we need the following.

Definition 4.22
Let G be a group and $H \leq G$. Then H is a *trivial intersection subgroup* (or *TI subgroup*) if for any $g \in G$ we have that $H \cap H^g = 1$ or $H \cap H^g = H$.

Theorem 4.23
Let $P \in \mathrm{Syl}_p\, G$ be a trivial intersection subgroup and $L = N_G(P)$. Then there is a one-to-one correspondence between isomorphism classes of non-projective indecomposable kG- and kL-modules such that if V and U are such modules, respectively, then

$$V_L \cong U \oplus Q \qquad \mathrm{Ind}_L^G\, U \cong V \oplus R$$

for Q and R projective kL- and kG-modules, respectively.

So the correspondence mentioned above is determined by induction or restriction. In fact, all projective indecomposable summands of Q and R are isomorphic.

Proof. By Mackey's theorem (Lemma 4.9) we have

$$(\mathrm{Ind}_L^G\, U)_L \cong \bigoplus_{s \in L\backslash G/L} \mathrm{Ind}_{L\cap L^s}^L\, s^{-1}(U)$$

so that $\mathrm{Ind}_L^G\, U$ has U as a direct summand corresponding to $s \in L$ and a sum of modules induced from $L \cap L^s$ for $s \notin L$. However, P is a normal (thus unique) Sylow p-subgroup of L with $P \cap P^s = 1$ for $s \notin L$ so that $P \cap P^s = 1 \in \mathrm{Syl}_p(L \cap L^s)$. So $L \cap L^s$ is a p'-group and so every $L \cap L^s$-module is projective. Thus $(\mathrm{Ind}_L^G\, U)_L \cong U \oplus Y$ where Y is projective since the induction of a projective module is itself projective.

Now, write $\mathrm{Ind}_L^G\, U = V_1 \oplus \cdots \oplus V_n$ as a direct sum of indecomposable modules. Since L contains a Sylow p-subgroup of G, by Corollary 4.14 we have that V_i is projective if and only if its restriction to L is. However, $(\mathrm{Ind}_L^G\, U)_L$ has a unique non-projective indecomposable summand in any decomposition into indecomposables, so all V_i bar one must be projective. Let the non-projective one be $V_1 \cong V$. So we now know that $\mathrm{Ind}_L^G\, U \cong V \oplus Q$ for Q projective and $V_L \cong U \oplus R$ for R projective since $U \oplus Y \cong (\mathrm{Ind}_L^G\, U)_L \cong V_L \oplus Q_L$.

Now, suppose that V is a non-projective indecomposable kG-module. As L contains a Sylow p-subgroup of G, every kG-module is relatively L-projective and in particular there exists an indecomposable kL-module U such that $V \mid \mathrm{Ind}_L^G\, U$ and V is not projective as its induction is not projective. Thus $V_L \cong U \oplus R$ for some projective R. We have thus provided a map from the set of isomorphism classes of non-projective indecomposable kG-modules to the same class for kL-modules, along with its inverse, as required. ∎

One important property of the above is as follows.

Corollary 4.24
Let V_1 and V_2 be non-projective indecomposable kG-modules with U_1 and U_2 the corresponding kL-modules. Then there is a non-split exact sequence

$$0 \to V_1 \to V \to V_2 \to 0$$

if and only if there is a non-split exact sequence

$$0 \to U_1 \to U \to U_2 \to 0.$$

Before returning to $\mathrm{SL}_2(p)$, we require one additional property of this correspondence and for this property, we need the following.

Definition 4.25
Let V_1 and V_2 be kG-modules. Then we define $\overline{\mathrm{Hom}}_G(V_1, V_2)$ to be the quotient of the vector space $\mathrm{Hom}_G(V_1, V_2)$ by the subspace of all homomorphisms which factor through a projective kG-module, where we say that a map factors through a projective if it is the composition of a homomorphism $U_1 \to P \to U_2$ with P projective.

Theorem 4.26
If V_1 and V_2 are non-projective indecomposable kG-modules and U_1 and U_2 are the corresponding kL-modules then

$$\overline{\mathrm{Hom}}_G(V_1, V_2) \cong \overline{\mathrm{Hom}}_L(U_1, U_2).$$

To ease some notation in the future, we shall expand one of our previous definitions.

Definition 4.27
Let V be a kG-module. Then we define the *projective cover* $\mathcal{P}(V)$ of V to be the minimal projective module with V as its quotient.

Note that we can do this since every module is a quotient of a free module, and thus a quotient of a projective module. If $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ is semisimple then $\mathcal{P}(V) \cong \mathcal{P}(V_1) \oplus \mathcal{P}(V_2) \oplus \cdots \oplus \mathcal{P}(V_n)$, and more generally for an arbitrary kG-module V we have that $\mathcal{P}(V) \cong \mathcal{P}(\operatorname{head} V)$.

Lemma 4.28
Suppose that every projective indecomposable kG-module is uniserial. Then every indecomposable kG-module is uniserial.

Proof. Suppose that M is an indecomposable kG-module with $V \leq M$ irreducible and let M' be the submodule of M which is maximal such that $M' \cap V = 0$. Then by maximality $\operatorname{soc}(M/M') = V$. By the dual of Lemma 3.20, M/M' is a submodule of $\mathcal{P}(V)$ and thus uniserial. Thus $\operatorname{head}(M/M')$ is simple so that $P \coloneqq \mathcal{P}(M/M')$ is a PIM thus uniserial. Now, the projective covering map $\pi \colon P \to M/M'$ may be lifted to a map $\pi' \colon P \to M$ whose image is a uniserial submodule of M containing V. As such, $M' \cap \pi'(P) = 0$ and so $M = M' \oplus \pi'(P)$. Since $\pi'(P) \neq 0$ and M is indecomposable, $M' = 0$ and we are done. ∎

Note that since every uniserial module is a quotient of the projective cover of its head by Lemma 3.20, this means that every indecomposable kG-module is a quotient of a projective in this case.

Lemma 4.29
Suppose that G has a cyclic normal Sylow p-subgroup. Then every projective indecomposable kG-module is uniserial.

Proof. Let $V \in \operatorname{Irr}_k G$, $Q \in \operatorname{Syl}_p G$ and $P := \mathcal{P}(V)$. Then P_Q is a projective, thus free, kQ-module of rank $\dim V$. By Lemma 3.23, the radical series of P and P_Q coincide and, since kQ has radical length $|Q|$ and each radical layer of P_Q has dimension $\dim V$, the same is true of P. Now, let $U := P/\operatorname{rad}^2 P$.

Let $V \cong k$ be the trivial module. By the above, $W := \operatorname{rad} P/\operatorname{rad}^2 P$ must also be a one-dimensional simple module. Let $U := P/\operatorname{rad}^2 P$. Then we have a non-split extension

$$0 \to W \to U \to k \to 0$$

with $\dim U = 2$. (In fact, there is a unique isomorphism class of such modules U since Q is cyclic, but we shall neither say nor see more of this here)

Now, the module $V \otimes U$ has an irreducible (as $\dim W = 1$) submodule $V \otimes W$ and quotient $V \otimes k \cong V$. Since U is not semisimple, by Lemma 3.23 for a generator x of Q we may choose nonzero $u \in (1-x)U = \operatorname{rad} U$. If for nonzero $v \in V$ we have that $(1-x)(v \otimes u)$ is nonzero then $\operatorname{rad}(V \otimes U) \neq 0$ and thus $V \otimes U$ is not semisimple. As $Q \trianglelefteq G$, V_Q is semisimple by Clifford's Theorem (Theorem 3.15), and since Q is a p-group we have that $V_Q^Q = V_Q$ so that $xv = v$. Thus

$$(1-x)(v \otimes u) = v \otimes u - x(v \otimes u) = v \otimes u - v \otimes xu = v \otimes (1-x)u \neq 0$$

as both v and $(1-x)u$ are nonzero. Thus $V \otimes U$ is itself a uniserial module of composition length two. It is easy to check that in fact $V \otimes U \cong P/\operatorname{rad}^2 P$. Now, suppose that M is a kG-module with head V. Since M is a quotient of P, either $M \cong V$ or $\operatorname{rad} M/\operatorname{rad}^2 M \cong V \otimes W$.

So, we know that the PIM P has composition length $|Q|$, its head is isomorphic to V and the second radical layer is $V \otimes W$. The radical $\operatorname{rad} P$ of P is thus a proper submodule of P with head $V \otimes W$ and composition length $|Q| - 1$. By the previous discussion, either $\operatorname{rad} P \cong V$ (and $|Q| = 2$) or $\operatorname{rad}^2 P/\operatorname{rad}^3 P \cong V \otimes W \otimes W$. Continuing in this way, we see that the i^{th} radical factor of P is isomorphic to the irreducible module $V \otimes (W^{\otimes(i-1)})$ and, in particular, P is uniserial. ■

Combining these two results, we obtain the following.

Corollary 4.30
Suppose that G has a cyclic normal Sylow p-subgroup. Then every indecomposable kG-module is uniserial.

Example 4.31
We return to our main example for this course, so once again let $G := \mathrm{SL}_2(p)$. Let P be the set of lower unitriangular matrices in G, so that $P \in \operatorname{Syl}_p G$. Since $|P| = p$, clearly P is a TI subgroup. Let $L := N_G(P)$, then L is the set of lower triangular matrices of determinant 1. Let $g := \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \in L$ and $\varphi\colon L \to \mathbb{F}_p^*$ be the map given by $\varphi(g) = a$. Then clearly φ is a homomorphism with kernel P and in particular L/P is cyclic of order $p-1$. As such, $\operatorname{Irr}_k L$ consists of $p-1$ 1-dimensional modules lifted from this quotient and by Corollary 4.30 all indecomposable kL-modules are uniserial.

Let U_j denote the 1-dimensional kL-module in which the element g acts as multiplication by a^j. Then U_{j_1} and U_{j_2} are isomorphic as kL-modules if and only if $j_1 \equiv j_2 \mod p-1$ and $U_{j_1} \otimes U_{j_2} \cong U_{j_1+j_2}$. Recall that V_2 is the natural module for G which we regard as having basis

X and Y with $gX = aX + cY$ and $gY = a^{-1}Y$. Then kY is an L-submodule of V_2 isomorphic to U_{-1} with quotient U_1 and it is easy to check that $(V_2)_L$ is indecomposable thus uniserial.

Combining this with the proof of Lemma 4.29, we see that the module W in our particular case may be taken to be U_{-2}. Thus, if M is an indecomposable kL-module with head $M \cong U_j$ then the radical factors of M are U_j, U_{j-2}, U_{j-4}, $\ldots$. In particular, the radical series of M is determined solely by head M and $\dim M \leq p$.

Back in Example 3.13 it was shown that $\operatorname{soc}((V_i)_P)$ was 1-dimensional with basis Y^{i-1} and since P lies in the kernel of all simple kL-modules this remains the socle of $(V_i)_L$. Looking at the action of L on this basis, we see that $\operatorname{soc}((V_i)_L) \cong U_{-(i-1)}$ and since $\dim V_i = i$ we obtain a picture

$$\begin{matrix} U_{i-1} \\ U_{i-3} \\ \vdots \\ U_{-i+1} \end{matrix}$$

of $(V_i)_L$ as the unique indecomposable kL-module of dimension i with head U_{i-1}.

We now require some lemmas before we can properly describe the structure of the kG-modules in this case.

Lemma 4.32
Let $1 \leq i < p-1$. Then there exists a non-split extension

$$0 \to V_{p-i-1} \to V \to V_i \to 0.$$

Proof. Let M be the indecomposable kL-module of dimension $p-1$ with head $M \cong U_{i-1}$. Then $M/\operatorname{rad}^i M$ has dimension i and head U_{i-1}, thus $M/\operatorname{rad}^i M \cong (V_i)_L$ and $\operatorname{head}(\operatorname{rad}^i M)$ is simply $\operatorname{soc}(M/\operatorname{rad}^i M) \otimes U_{-2} \cong U_{-i-1} \cong U_{(p-1-i)-1}$. But $\dim \operatorname{rad}^i M = (p-1) - i$ and so $\operatorname{rad}^i M \cong (V_{p-1-i})_L$ and so we have an extension

$$0 \to (V_{p-1-i})_L \to M \to (V_i)_L \to 0$$

which cannot split as M is indecomposable and so we are done by Corollary 4.24. ■

Lemma 4.33
Let $1 < i \leq p-1$. Then there exists a non-split extension

$$0 \to V_{p+1-i} \to V \to V_i \to 0.$$

Proof. Let $R_i := \mathcal{P}(U_i)$ and set $M := R_{i-1} \oplus U_{-i+1}$, $W := \operatorname{rad}^{i-1}(R_{i-1})$. Then $W \cong (V_{p+1-i})_L$. Let $\varphi\colon W \twoheadrightarrow U_{-i+1}$ and let $Z := \{(w, \varphi(w)) \mid w \in W\} \leq W \oplus U_{-i+1}$. It is easy to see that $Z \cong W$, and we claim that $M/Z \cong (V_i)_L$. As the dimensions match up, it is sufficient to show that $\operatorname{head}(M/Z) \cong U_{i-1}$. Since $\operatorname{rad}(M/Z) = (\operatorname{rad} M + Z)/Z$, $\operatorname{rad} M + Z \leq M$ and $(R_{i-1} \oplus U_{-i+1})/(\operatorname{rad}(R_{i-1}) \oplus U_{-i+1}) \cong U_{i-1}$ we need only show that $\operatorname{rad}(R_{i-1}) \oplus U_{-i+1} \leq \operatorname{rad} M + Z$. It is also clear that $\operatorname{rad}(R_{i-1}) \leq \operatorname{rad} M$ thus we reduce to requiring that $U_{-i+1} \leq \operatorname{rad} M + Z$.

If $u \in U_{-i+1}$, choose $w \in W$ with $\varphi(w) = u$. Thus $(0, u) = (w, u) - (w, 0) \in Z + W$, but $W \leq \operatorname{rad} M$ so $M/Z \cong (V_i)_L$ as claimed and we obtain an extension

$$0 \to (V_{p+1-i})_L \to M \to (V_i)_L \to 0$$

which cannot split since $M \cong R_{i-1} \oplus U_{-i+1}$ is a decomposition into indecomposable summands and clearly not isomorphic to $(V_{p+1-i})_L \oplus (V_i)_L)$. We are thus again done by Corollary 4.24. ■

With these lemmas in place, we may now explore the structure of the PIMs $P_1, \ldots, P_p$ for $p > 2$. By Lemma 4.32 there is a uniserial module with radical factors V_1, V_{p-2} which is thus a quotient of P_1 and so we automatically know that P_1 looks like

$$\begin{matrix} V_1 \\ V_{p-2} \\ X \\ V_1 \end{matrix}$$

for some (possibly zero) module X. In particular $\dim P_1 \geq 1 + p - 2 + 1 = p$ with equality if and only if $\mathcal{H}(P_1) \cong V_{p-2}$ (equivalently, $X = 0$).

Similarly, by Lemma 4.33, $\dim P_{p-1} \geq 2p$ with equality if and only if $\mathcal{H}\, P_{p-1} \cong V_2$. Now suppose that $1 < i < p - 1$. Using Lemmas 4.32 and 4.33 we see that there are submodules M_+, M_- of $\operatorname{rad} P_i$ such that $\operatorname{rad} P_i/M_+ \cong V_{p+1-i}$ and $\operatorname{rad} P_i/M_- \cong V_{p-1-i}$. Letting $M = M_+ \cap M_-$ we have $\operatorname{rad} P_i/M \cong V_{p+1-i} \oplus V_{p-1-i}$ and so we obtain a picture

$$\begin{matrix} V_i \\ V_{p+1-i} \oplus V_{p-1-i} \\ X \\ V_i \end{matrix}$$

for some (possibly zero) module X. Again we have that $\dim P_i \geq 2p$ with equality if and only if $X = 0$ above. Moreover,

$$\begin{aligned} p(p^2 - 1) &= \dim kG \\ &= \sum_{i=1}^{p} \dim V_i \dim P_i \\ &\geq p + 2p \sum_{i=2}^{p-2} i + (p-1)2p + p^2 \\ &= p^3 - p \end{aligned}$$

and so we must have equality throughout and the modules X in the above pictures are zero. Thus the projective modules for G are, shockingly, the same as we showed in Example 3.38. If we did not already know that $V_1, \ldots, V_p$ were the only irreducible modules, investigating the above chain of (in)equalities would have also shown this.

Green Correspondence

Theorem 4.23 from the last section is a special case of the Green correspondence which is much easier to apply. We will now introduce the full version of the Green correspondence, though this is much more complicated and requires much more care to state correctly. Let Q be a p-subgroup of G and $L \coloneqq N_G(Q)$. Through the Green correspondence we will obtain a one-to-one correspondence between indecomposable kG-modules with vertex Q and indecomposable kL-modules with the same vertex, and this correspondence will be given via induction and restriction though it will not be as friendly as in the trivial intersection case.

We now instead let L be any subgroup of G which contains $N_G(Q)$. For P, $R \leq G$, we will use the notation $P \leq_G R$ to mean that some conjugate of P is contained in R, so P is a subgroup of R up to conjugacy. If $\mathcal{H}$ is a collection of subgroups of G then we say that $P \leq_G \mathcal{H}$ if there exists $H \in \mathcal{H}$ such that $P \leq_G H$. Now, fix the below collections of subgroups of G.

$$\begin{aligned}
\mathfrak{x} &:= \{Q^s \cap Q \mid s \in G,\ s \notin L\},\\
\mathfrak{n} &:= \{Q^s \cap L \mid s \in G,\ s \notin L\},\\
\mathfrak{z} &:= \{R \mid R \leq Q,\ R \not\leq_G \mathfrak{x}\}.
\end{aligned}$$

One should think of the collections $\mathfrak{x}$ and $\mathfrak{n}$ as 'small' subgroups of Q while $\mathfrak{z}$ should be thought of as consisting of large subgroups. Each element of $\mathfrak{x}$ is proper in Q since $s \notin L$ and L contains $N_G(Q)$, and also $Q \in \mathfrak{z}$.

Also, for a collection $\mathcal{H}$ of subgroups of G we say that a kG-module V is relatively $\mathcal{H}$-projective if each indecomposable summand of U is relatively projective for some subgroup in $\mathcal{H}$.

Now, the Green correspondence itself is as follows.

Theorem 4.34 (Green Correspondence)
There is a one-to-one correspondence between isomorphism classes of indecomposable kG-modules with vertex in $\mathfrak{z}$ and isomorphism classes of indecomposable kL-modules with vertex in $\mathfrak{z}$. If U and V are corresponding such modules for L and G, respectively, then U and V have the same vertex and

$$V_L \cong U \oplus Y \qquad \textit{and} \qquad \mathrm{Ind}_L^G U \cong V \oplus X$$

where Y is a relatively $\mathfrak{n}$-projective kL-module and X is a relatively $\mathfrak{x}$-projective kG-module.

Of course, we have repeatedly claimed that Theorem 4.23 is a special case of this result, so let's verify that claim. Suppose Q is a TI subgroup. Then $\mathfrak{x} = \{1\}$ as $Q^s \cap Q = 1$ for all $s \notin L$, and $\mathfrak{z}$ is the set of all nontrivial subgroups of Q. Further, if $Q \in \mathrm{Syl}_p G$ then $Q^s \cap L = 1$ if $s \notin L$, otherwise $Q^s \cap L$ is a p-subgroup of L and thus there is $x \in L$ such that $(Q^s \cap L)^x \leq Q$ and so $Q^x s \cap Q \neq 1$ and $xs \notin L$. Thus $\mathfrak{n} = 1$ and so X and Y are projective.

Now, before proving the Green correspondence we have a few other results to prove.

Lemma 4.35
Let $R \leq Q$. Then the following are equivalent.

i) $R \leq_G \mathfrak{x}$,

ii) $R \leq_L \mathfrak{x}$,

iii) $R \leq_L \mathfrak{n}$.

Proof. Suppose i) holds. Then there is some $g \in G$ such that $R^g \leq Q \cap Q^s$ for some $s \in G \setminus L$. If $g \in L$ clearly we have ii), but if $g \notin L$ then $R \leq Q^{g^{-1}}$ and so $R \leq Q \cap Q^{g^{-1}}$. But then $R \in \mathfrak{x}$ and so clearly $R \leq_L \mathfrak{x}$ and ii) holds.

Now suppose ii) holds. Then there exists $x \in L$ and $s \in G \setminus L$ such that $R^x \leq Q \cap Q^s$. But then clearly $R^x \leq L \cap Q^s$ and $R \leq_L \mathfrak{n}$ so iii) holds. Finally, if iii) holds we have some $x \in L$ and $s \in G \setminus L$ such that $R^x \leq L \cap Q^s$ and so $R \leq L \cap Q^{x^{-1}s}$, but $x^{-1}s \notin L$ and so $R \leq_G \mathfrak{x}$ and thus i) holds. ■

Lemma 4.36
Suppose V is a relatively $\mathfrak{x}$-projective kG-module. Then V_L is relatively $\mathfrak{n}$-projective. Similarly, if U is a relatively Q-projective and a relatively $\mathfrak{n}$-projective kL-module then $\mathrm{Ind}_L^G U$ is relatively $\mathfrak{x}$-projective.

Proof. Let $W \mid V$ be indecomposable so that W is relatively projective for some subgroup of the form $Q^s \cap Q$ for $s \notin L$. Then by Mackey's theorem we have that W_L is relatively projective for the collection of subgroups of the form

$$(Q^s \cap Q)^t \cap L = Q^{ts} \cap Q^t \cap L.$$

But either $t \notin L$ or $ts \notin L$ and so such a subgroup is contained in some element of $\mathfrak{n}$ and so W_L and V_L are relatively $\mathfrak{n}$-projective. Conversely, if $W \mid U$ is indecomposable with vertex P then $P \leq_L \mathfrak{n}$ and so $\mathrm{Ind}_L^G W$ is relatively P-projective and $P \leq_G \mathfrak{z}$ by the previous lemma, so $\mathrm{Ind}_L^G W$ is relatively $\mathfrak{x}$-projective and we are done. ■

All that remains is to prove the theorem itself. We first prove two more lemmas.

Lemma 4.37
If V is an indecomposable kG-module with vertex $R \in \mathfrak{z}$ then $V_L \cong U \oplus Y$ where U is an indecomposable kL-module with vertex R, V is a summand of $\mathrm{Ind}_L^G U$ and Y is relatively $\mathfrak{n}$-projective.

Proof. By Lemma 4.18, there is an indecomposable kL-module U with vertex R such that $V \mid \mathrm{Ind}_L^G U$. By Lemma 4.19 we have that $(\mathrm{Ind}_L^G U)_L \cong V \oplus Y_1$ for some relatively $\mathfrak{n}$-projective module Y_1 and so there is some summand Y of Y_1 such that either $V_L \cong U \oplus Y$ or $V_L \cong Y$. Again using Lemma 4.18 we have that there is some indecomposable summand W of V_L with vertex R. Suppose $W \mid Y$, then we must have that $R \leq_L \mathfrak{n}$ and $R \leq_G \mathfrak{x}$ by Lemma 4.35 and in particular $R \notin \mathfrak{z}$, a contradiction. Thus $W \cong U$ and $V_L \cong U \oplus Y$ as required. ■

Lemma 4.38
Suppose U is an indecomposable kL-module with vertex $R \in \mathfrak{z}$. Then $\mathrm{Ind}_L^G U \cong V \oplus X$ where V is an indecomposable kG-module with vertex R such that $U \mid V_L$ and X is relatively $\mathfrak{x}$-projective.

Proof. Take a direct sum decomposition $\mathrm{Ind}_L^G U = U_1 \oplus \cdots \oplus U_r$. Since $(\mathrm{Ind}_L^G U)_L \cong U \oplus Y$ for Y relatively $\mathfrak{n}$-projective by Lemma 4.19, relabeling our modules if necessary, we have that $(V_1)_L \cong U + Y_1$ and $(V_i)_L \cong Y_i$ for $2 \leq i \leq r$ for kL-modules Y_i such that $Y \cong Y_1 \oplus \cdots \oplus Y_r$. We claim V_1 has a vertex in $\mathfrak{z}$ and that the remaining V_i are relatively $\mathfrak{x}$-projective. Clearly V_1 has a vertex in $\mathfrak{x}$ since $V_1 \mid \mathrm{Ind}_L^G U$ and V_1 is not relatively $\mathfrak{x}$-projective by Lemma 4.36.

Let $V = V_1$ and $X = V_2 + \cdots + V_r$. Then $V \mid \mathrm{Ind}_L^G U$ and $\mathrm{Ind}_L^G U \cong V \oplus X$ for X relatively $\mathfrak{x}$-projective. The only remaining claim is that V has vertex R. We know that a vertex of U lies in $\mathfrak{z}$ and so the previous lemma tells us that in any decomposition of V_L into indecomposable summands there is a unique summand which is not relatively $\mathfrak{n}$-projective and that this summand has a vertex equal to a vertex of V. But $V_L \cong U \oplus Y$ and so V has vertex R since U does. ■

Proof of Theorem 4.34. The majority of the theorem was proved by the previous two lemmas, all that remains is to show that this correspondence is in fact one-to-one. This boils down to two claims: First, that if V is an indecomposable kG-module with vertex $R \in \mathfrak{z}$ as well as U is as in Lemma 4.37 and $\mathrm{Ind}_L^G U \cong V' \oplus Y$ as in Lemma 4.38 then $V \cong V'$, and a similar result for if we instead start with U. However, in Lemma 4.37 it was shown that $V \mid \mathrm{Ind}_L^G U$ and in Lemma 4.38 that $U \mid V_L$ and thus we are done. ■

A theorem that we shall use in a later proof which requires the Green correspondence to prove is below.

Theorem 4.39
Let V be an indecomposable kG-module with vertex Q and W be the corresponding kL-module. If M is a kG-module then $V \mid M$ if and only if $W \mid M_L$ and if M is an indecomposable kG-module with $W \mid M_L$ then $M \cong V$.

Proof. This is [1, Theorem 12.2]. ■

5 Blocks

Up until this point, we have broadly studied the structure of kG-modules in isolation, referring only to the structure we can deduce ourselves or from investigating p-local subgroups. However, indecomposable modules can be more strongly related to one another than this. For a brief moment, we shall state a result about arbitrary (associative, unital) algebras again.

Theorem 5.1
Let A be an algebra. Then A has a unique decomposition into a sum of indecomposable subalgebras $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$.

These indecomposable summands A_i are called the *blocks* of A with each one a two-sided ideal of A. Addition and multiplication across A is in fact componentwise between these blocks, so if $a_i \in A_i$ and $a_j \in A_j$ then $a_i a_j \in A_i \cap A_j = \delta_{ij} A_j$ (where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise).

Proof. Take a decomposition of $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ into indecomposable subalgebras and let B be a summand in any other such decomposition with $b = a_1 + a_2 + \cdots + a_n \in B$ for $a_i \in A_i$. Since each A_i is unital and B is an ideal, we have $a_i \in B$ for each i. Hence we have a decomposition $B = (B \cap A_1) \oplus (B \cap A_2) \oplus \cdots \oplus (B \cap A_n)$ with each $B \cap A_i$ an ideal of B. As B is indecomposable, only one of these summands is nonzero, so $B \subseteq A_i$ for some i. Applying an identical argument to this A_i with regards to the decomposition of A from which we obtained B, we see that in fact $B = A_i$ for some i and thus our decomposition is unique. ■

For now, fix the notation $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ as a decomposition of the algebra A into blocks. Take an A-module M. If $A_i M = M$ and $A_j M = 0$ for all $j \neq i$ then we say that M *lies in the block* A_i, and given an A_i-module N we may easily regard N as an A-module by ensuring that $A_j N = 0$ for all $j \neq i$. This clearly yields all A-modules lying in the block A_i.

Take a decomposition $1 = e_1 + e_2 + \cdots + e_n$ for $e_i \in A_i$ the identity element. If M lies in A_i then e_i is the identity on M and $e_j M = 0$ for all $j \neq i$, and again the converse also holds since $AM = Ae_iM = A_iM$. Following this to its logical conclusion, we also observe that submodules, quotients and direct sums of A-modules lying in the block A_i also lie in A_i. Further, if $M_i \in A_i$ and $M_j \in A_j$ with $i \neq j$ then $\operatorname{Hom}_A(M_i, M_j) = 0$ since for any map $\varphi\colon M_i \to M_j$ we have that $e_i\varphi(M_i)$ is the identity but $e_iM_j = 0$. These general ideas suggest the following result.

Theorem 5.2
Let M be an A-module. Then M has a unique decomposition

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

where $M_i \in A_i$ for each i.

Proof. We take the obvious decomposition by letting $M_i := A_iM$, this clearly yields a direct sum decomposition of M so it remains only to check uniqueness. Suppose that $M = N_1 \oplus \cdots \oplus N_n$ is another such decomposition with $N_i \in A_i$. Then $N_i \leq A_iN_i \leq A_iM = M_i$ so $N_i \leq M_i$ and hence $N_i = M_i$ since these N_i sum to M. ■

So, given an A-module M lying in the block A_i, as every submodule and quotient of M lies in A_i we also have that every composition factor of M lies in A_i. Conversely, if every composition factor of a module M lies in the same block A_i, M lies in that block: e_i acts as the identity on every factor and e_j annihilates every factor, and it is easy to see that the same thus holds for the entire module.

We thus reduce to the question: Given two simple A-modules S and T, when do S and T lie in the same block? Turns out, someone already asked this question.

Proposition 5.3
Let S and T be simple A-modules. Then the following are equivalent.

i) *S and T lie in the same block.*

ii) *There are simple A-modules $S = S_1, S_2, S_3, \ldots, S_m = T$ such that for each i we have S_i and S_{i+1} are both composition factors of an indecomposable projective A-module.*

iii) *There are simple A-modules $S = T_1, T_2, T_3, \ldots, T_m = T$ such that T_i and T_{i+1} are equal or there is a non-split extension of one by the other.*

In the case where A is an algebra and not a group ring, technically some of the results that we use in this proof have not been given in this course. You can either believe that they still hold in this case, or just replace A by kG throughout.

The following lemma will feature in the proof:

Lemma 5.4
If X is an A-module and $\operatorname{rad} X$ is semisimple then for any composition factor W of $\operatorname{rad} X$ there is a composition factor U of $\operatorname{head} X$ and a non-split extension of U by W.

Proof. Let $\operatorname{rad} X = W \oplus W'$. Then the module X/W' still satisfies the hypothesis above and so without loss of generality we may assume that $\operatorname{rad} X = W$. There are submodules $Y_1, \ldots, Y_n$ of X containing W such that X/W is the direct sum of the simple modules Y_i/W. Now, $\operatorname{rad} X = (\operatorname{rad} A)X$ and so $\operatorname{rad} X$ is the sum of all $(\operatorname{rad} A)Y_i = \operatorname{rad} Y_i$. Hence there is some j such that $\operatorname{rad} Y_j \neq 0$. Since $W = \operatorname{rad} X$ is simple, we have that $\operatorname{rad} Y_j = W$ and so Y_j is a non-split extension of the simple module Y_j/W by W. ■

Proof of Proposition 5.3. Suppose that T and T' are simple A-modules and we have a non-split extension M of T by T'. Then M is uniserial and thus a quotient of the indecomposable projective module corresponding to T, and so iii) implies ii). Now, by Theorem 5.2 we see that any indecomposable A-module lies in a block and thus ii) implies i). The hard part is getting back to iii) from here.

Let S and T be simple A-modules lying in a block B of A and suppose that S and T are not related as in ii). So we may take a decomposition

$$B = P_1 \oplus P_2 \oplus \cdots \oplus P_s \oplus Q_1 \oplus \cdots \oplus Q_t$$

into projective modules where a composition factor of each P_i is related to S via ii) and no composition factor of any Q_i is. Then $\operatorname{Hom}_B(P_i, Q_j) = \operatorname{Hom}_B(Q_j, P_i) = 0$ for all i, j as no composition factor of any Q_j may be a composition factor of any P_i. Then $P = P_1 \oplus P_2 \oplus \cdots \oplus P_s$ and $Q = Q_1 \oplus \cdots \oplus Q_t$ are invariant under all endomorphisms of B and are in fact ideals, contradicting the indecomposability of B as an algebra. So i) implies ii).

It remains only to show that ii) implies iii). It is sufficient to show that if V is a simple A-module with corresponding projective indecomposable module P then V is related to every composition factor of P as in iii). More specifically, we claim that if W is a composition factor of $\operatorname{rad}^{i+1} P / \operatorname{rad}^{i+2} P$ then there is a composition factor U of $\operatorname{rad}^i P / \operatorname{rad}^{i+1} P$ and a non-split extension of U by W. In fact, this follows from Lemma 5.4 applied to $\operatorname{rad}^i P / \operatorname{rad}^{i+2} P$. ■

We now take a brief diversion to look at $\mathrm{SL}_2(p)$ again.

Example 5.5
Let $G = \mathrm{SL}_2(p)$ with simple modules $V_1, \ldots, V_p$. Recall that the module V_p is projective and that the composition factors of each projective indecomposable kG-module are either all of even dimension or all of odd dimension. Provided $p \neq 2$, referring to Example 3.38 we can immediately see that $\operatorname{Irr}_k G = B_1 \sqcup B_2 \sqcup \{V_p\}$ where $B_1 = \{V_1, V_3, \ldots, V_{p-2}\}$ and $B_2 = \{V_2, V_4, \ldots, V_{p-1}\}$ is a decomposition of $\operatorname{Irr}_k G$ into blocks. When $p = 2$, we must have $B_2 = 0$.

Example 5.6
The second condition of Proposition 5.3 also tells us how to decompose A into blocks if we already know a decomposition of A into projective indecomposable modules. Suppose that $A = P_1 \oplus \cdots P_n$ for P_i projective. Suppose that $P_1, \ldots, P_j$ all lie in the same block and that no other P_i lie in this block. Then the block is equal to $P_1 \oplus \ldots \oplus P_j$.

We now wish to specialise our results to the case of group algebras once again. Regard the group algebra kG as a module for the group algebra $k[G \times G]$ via the action $(g_1, g_2)h = g_1 h g_2^{-1}$ (note the inverse is on the right as we require this to be a left action since we work with left modules. Inconveniently, I use $h^g := g^{-1}hg$ so that we have to conjugate by g^{-1} to obtain a left action).

Then submodules of kG as a $k[G \times G]$-module are precisely the two-sided ideals of kG and so in particular kG has a unique decomposition into indecomposable $k[G \times G]$-modules and these are the blocks of kG. As $k[G \times G]$-modules, these blocks are pairwise non-isomorphic since their annihilators in $k[G \times 1] \leq k[G \times G]$ are different.

Let $\delta \colon G \to G \times G$ be the diagonal embedding: $\delta(g) := (g, g)$.

Theorem 5.7
Let B be a block of kG. Then, as a $k[G \times G]$-module, B has a vertex of the form δD for D a p-subgroup of G.

Proof. It is sufficient to show that B is relatively $\delta(G)$-projective, since then $\delta(G)$ contains a vertex of B which must be of the required form. In fact, since $B \mid kG$ as $k[G \times G]$-modules we need only show that kG is $\delta(G)$-projective. But kG contains the k-subspace spanned by 1, which is a trivial $k[\delta(G)]$-module that we shall denote by k. Further,

$$\dim kG = |G| \dim k = [G \times G : \delta(G)] \dim k$$

and clearly k generates kG as a $k[G \times G]$-module. Hence by Corollary 4.4 we have that $kG \cong \operatorname{Ind}_{\delta(G)}^{G \times G} k$ and kG is relatively $\delta(G)$-free and thus in particular it is relatively $\delta(G)$-projective. ∎

Now, if H, $K \leq G$ are such that $\delta(H)$ and $\delta(K)$ are conjugate in $G \times G$ then H and K are conjugate in G. This allows us to make the following definition.

Definition 5.8
Let B be a block of kG with vertex (as a $k[G \times G]$-module) $\delta(D)$. Then the conjugacy class of subgroups D^G are the *defect groups* of B. If $|D| = p^d$ then we say that B has *defect d*.

From time to time we will abuse notation by referring to *the* defect group of a block to refer to any choice of subgroup from the conjugacy class. In the same way that the size of the vertex of a module measures its distance from projectivity, the size of the defect group of a block B measures its distance from being semisimple as an algebra. If a block B has a Sylow p-subgroup of G as a defect group, then we say that B is of *maximal defect*.

Now, much as the modules in a block are related to one another, the defect group of a block also relates to the modules within that block.

Theorem 5.9
Let B be a block of G with defect group D. Then any indecomposable module lying in B has a vertex contained in D.

Proof. Regard B as a kG-module with G acting by conjugation. If V is a kG-module then there is a map $\varphi \colon B \otimes V \to V$ given by $\varphi(\beta \otimes v) = \beta v$ which can be seen to be a kG-homomorphism. We also have a homomorphism $V \to B \otimes V$: Let e be the unit element of B, then we map $v \in V$ to $e \otimes v$. Thus $V \mid B \otimes V$ and so if $B \otimes V$ is relatively D-projective then so is V. In fact, if B is relatively D-projective then so is $B \otimes V$ for any kG-module V.

We may regard B as a $k[\delta(G)]$-module via the isomorphism $G \cong \delta(G)$ and thus reduce to the requirement that B is relatively $\delta(D)$-projective as a $k[\delta(G)]$-module. However, B, as a $k[G \times G]$-module, is induced from a $k[\delta(G)]$-module and so, by Mackey's Theorem (Lemma 4.9) we have that, as a $k[\delta(G)]$-module, B is relatively projective for the subgroups $\delta(G) \cap \delta(D)^{(g_1,g_2)}$ for g_1, g_2 running over G. If every such subgroup is conjugate in $\delta(G)$ to a subgroup of $\delta(D)$ then we are done.

If $d \in D$, g_1, $g_2 \in G$ and $\delta(d)^{(g_1,g_2)} \in \delta(G)$, then $d^{g_1} = d^{g_2}$ so that

$$\delta(d)^{(g_1,g_2)} = (d^{g_1}, d^{g_2}) = (d^{g_1}, d^{g_1}) = \delta(d)^{\delta(g_1)}$$

is contained in a $\delta(G)$-conjugate of $\delta(D)$, as required. ■

Much as there is an irreducible module which we can construct for any group, the trivial module, there is a block that we can construct for any group too.

Definition 5.10
The *principal block* $b_0(G)$ of a finite group G is the block of kG which contains the trivial module.

Note by previous discussions, the trivial module k has the Sylow p-subgroups of G as its vertices and by Theorem 5.9 the order of a vertex of k is a lower bound on the order of a defect group of $b_0(G)$. As such, $b_0(G)$ is always a block of maximal defect for any group G, and it has the Sylow p-subgroups of G as its defect groups.

One might assume from the definition of a defect group that they can be an arbitrary p-subgroup of G, but this is in fact not the case. Let $O_p(G)$ denote the largest normal p-subgroup of G.

Theorem 5.11
Let B be a block of G with defect group D. Then

i) If $P \in \mathrm{Syl}_p G$ contains D then there exists $c \in C_G(D)$ such that $D = P \cap P^c$.

ii) We have $O_p(G) \leq D$, i.e. *D contains every normal p-subgroup of G.*

iii) We have $D = O_p(N_G(D))$.

The first statement implies the other two: Suppose i) holds and let $N \trianglelefteq G$ be a p-subgroup. Then N is contained in every Sylow p-subgroup of G and thus lies in the intersection of all of them. In particular, N lies in the intersection of any two, and thus in any defect group.

Now let $T \in \mathrm{Syl}_p(N_G(D))$ and choose $P \in \mathrm{Syl}_p(G)$ containing T. Then $D \leq T \cap T^c \leq P \cap P^c = D$. But $T^c \in \mathrm{Syl}_p(N_G(D))$ and so D is also the intersection of Sylow p-subgroups of $N_G(D)$, and since every normal p-subgroup of $N_G(D)$ lies in all Sylow p-subgroups of $N_G(D)$ we also obtain iii). Proving i) is much more difficult, and shall not be included in the lectures but we shall leave it here for completeness.

We first collect in this lemma some useful results.

Lemma 5.12
Let $H \leq G$ and $t \in G$.

i) The $k[H \times H]$-module $kHtH = \mathrm{Ind}_{\delta(H \cap H^{t^{-1}})^{(1,t)}}^{k[H \times H]} k$.

ii) If H is a p-group then $kHtH$ is indecomposable as a $k[H \times H]$-module with vertex $\delta(H \cap H^{t^{-1}})^{(1,t)}$.

iii) If $Q \leq H$ is a p-subgroup, $C_G(Q) \leq H$ and $t \notin H$ then no indecomposable summand of $kHtH$ has a vertex containing $\delta(Q)$.

The module $kHtH$ mentioned above is simply the k-span of the double coset HtH.

Proof. Consider the subgroup of $H \times H$ of elements which fix t. We have $(h_1, h_2)t = t$ if and only if $h_1 t h_2^{-1} = t$, that is $h_2 = h_1^t$. Thus this subgroup consists of all pairs (h, h^t) with $h \in H \cap H^{t^{-1}}$. More specifically, our subgroup is $\delta(H \cap H^{t^{-1}})^{(1,t)}$. Now, $\dim kHtH$ is the product of $|H|$ and the number of cosets in HtH, which is

$$\dim kHtH = |H|[H : H \cap H^{t^{-1}}] = [H \times H : \delta(H \cap H^{t^{-1}})^{(1,t)}.$$

Now, t generates $kHtH$ as a $k[H \times H]$-module and so i) follows and ii) also follows since it is induced from the trivial module of a p-subgroup, and since the trivial module has the Sylow p-subgroups as its vertices we obtain ii).

To see iii), from i) we have that each indecomposable summand of $kHtH$ has vertex contained in $\delta(H \cap H^{t^{-1}})^{(1,t)}$. If iii) were false, there would exist an $H \times H$-conjugate of $\delta(Q)$ which is contained in $\delta(H \cap H^{t^{-1}})^{(1,t)}$. In particular, there would exist h_1, $h_2 \in H$ such that $\delta(Q)^{(h_1,h_2)} \leq \delta(G)^{(1,t)}$ or, alternatively, $\delta(Q)^{(h_1,h_2)(1,t)^{-1}} \leq \delta(G)$. Thus if $x \in Q$ we have $x^{h_1} = x^{h_2 t^{-1}}$ and so $h_1 t h_2^{-1} \in C_G(Q)$ and $t \in h_1^{-1} C_G(Q) h_2 \leq H$, a contradiction. ∎

Proof of Theorem 5.11. From the discussion after the theorem statement, we see we only need to prove i). Since $\delta(D) \leq P \times P$ and B as a $k[G \times G]$-module has $\delta(D)$ as a vertex, by Lemma 4.18, $B_{P \times P}$ has an indecomposable summand with vertex $\delta(D)$. But $B \mid kG$ and $(kG)_{P \times P} \cong \bigoplus_{t \in P \backslash G/P} kPtP$ since G is a disjoint union of double cosets PtP. But Lemma 5.12 shows that each of these double cosets is indecomposable as a $k[P \times P]$-module with vertex $\delta(P \cap P^{t^{-1}})^{(1,t)}$. Hence there is some $t \in G$ such that $\delta(D)$ is $P \times P$-conjugate to $\delta(P \cap P^{t^{-1}})^{(1,t)}$. So there exists q, $r \in P$ such that $\delta(D) = \delta(P \cap P^{t^{-1}})^{(1,t)(q,r)^{-1}}$. In particular, $|D| = |P \cap P^{t^{-1}}|$. Further, we have $\delta(D)^{(q,r)(1,t)^{-1}} \leq \delta(G)$ and so for any $d \in D$ we have $d^q = d^{rt^{-1}}$ and so $qtr^{-1} \in C_G(D)$.

Set $c := qtr^{-1}$. It is thus sufficient to show that $D = P \cap P^{c^{-1}}$. This intersection clearly contains D since $D \leq P$ and $c \in C_G(D)$. Now,

$$\begin{aligned} |P \cap P^{c^{-1}}| &= |P \cap qtr^{-1} P rt^{-1} q^{-1}| \\ &= |P^q \cap P^{t^{-1}}| \\ &= |P \cap P^{t^{-1}}| \\ &= |D| \end{aligned}$$

and thus these two subgroups are equal, as required. ∎

Definition 5.13
Let $H \leq G$ and suppose that b is a block of H and B a block of G. We say that B *corresponds*

to b and write $B = b^G$ provided that, as $k[H \times H]$-modules, we have $b \mid B$ and that B is the only block with this property.

This yields a map from a subset of the blocks of H to the blocks of G. If such a block $B = b^G$ exists for a block b of H, we say that b^G is defined.

Lemma 5.14
Let b be a block of $H \leq G$ with defect group D. Then the following hold.

i) If b^G is defined then D is contained in a defect group of b^G.

ii) If $H \leq K \leq G$ while b^K, $(b^K)^G$ and b^G are defined then $b^G = (b^K)^G$.

iii) If $C_G(D) \leq H$ then b^G is defined.

Proof. Let E be a defect group of $B = b^G$ so that the $k[G \times G]$-module B has vertex $\delta(E)$. Since $b \mid B_{H\times H}$, Mackey's Theorem (Lemma 4.9) tells us that the vertex $\delta(D)$ of b is $G \times G$-conjugate to a subgroup of $\delta(E)$. However, if $(g_1, g_2) \in G \times G$ is such a conjugating element, then $D^{g_1} \leq E$ and this proves i). Part ii) then follows immediately from the definition since $b \mid (b^G)_{H\times H}$ and $b \mid ((b^K)^G)_{H\times H}$.

To show iii), it is sufficient to check that b occurs only once in a decomposition of $(kG)_{H\times H}$ into indecomposables. Now, kH is a direct sum of the blocks of H, which are pairwise nonisomorphic as $k[H \times H]$-modules. It thus siffices to show that $b \nmid kHtH$ for $t \notin H$. But $\delta(D)$ is a vertex of b and Lemma 5.12 says that no indecomposable summand of $kHtH$ has a vertex containing $\delta(D)$ and so we are done. ■

Note that the condition iii) of Lemma 5.14 is a sufficient but not a necessary condition. To see this, we have the following example.

Example 5.15
Let $G \coloneqq D_{2n}$ be dihedral of order $2n$ for n odd and let $p = 2$. Let H be the normal 2-complement in G, *i.e.* the index 2 normal subgroup which is cyclic of order n. Then H is an abelian $2'$-group and thus $|\mathrm{Irr}_k H| = |H| = n$. Let $b_0(G)$ denote the block of kG which contains the trivial module k. Since H is a $2'$-group, it is clear that $b_0(H) \cong \mathcal{P}(k) \cong k$. Since k is projective as a H-module, so is $\mathrm{Ind}_H^G k$, and since $\mathrm{Ind}_H^G k$ has dimension two it is either semisimple or a non-split extension of k by k. By previous discussions, the trivial module is projective precisely when kG is semisimple, which we know not to be the case here. As such, as a kG-module we have $\mathcal{P}(k)$ is a non-split extension of k by k and thus by condition ii) of Proposition 5.3 we have that $b_0(G) \cong \mathcal{P}(k)$ and the trivial module is the only irreducible module lying in the principal block.

Thus $b_0(H) \mid b_0(G)$ as $k[H \times H]$-modules. Clearly there can be no other block of G with this property, so $b_0(H)^G = b_0(G)$. Now, the defect group of the principal block for H is a Sylow 2-subgroup of H, but H is a 2'-group and thus the defect group for $b_0(H)$ is 1. But $C_G(1) = G$ is not contained in H.

Letting $n = 3$, we have $G \cong D_6 \cong \mathrm{SL}_2(2)$ and so this may actually be visualised through our previous examples.

We now wish to investigate how blocks of our group G relate to those of its subgroups. The situation here is not as simple as it is for modules, but there are still things that may be done.

The theorem we require the above result for is Brauer's First Main Theorem, below.

Theorem 5.16 (Brauer's First Main Theorem)
If $D \leq G$ is a p-subgroup and $N_G(D) \leq H$ then there is a one-to-one correspondence between the blocks of H with defect group D and the blocks of G with defect group D given by letting the block b of H correspond to the block b^G of G.

Proof. Note that since certainly $C_G(D) \leq H$ the block b^G is defined by Lemma 5.14. Let b be a block of H with defect group D and $B := b^G$. Since $H \times H \geq N_{G\times G}(\delta(D))$, b is indecomposable as a $k[H \times H]$-module, has vertex $\delta(D)$ and $b \mid B_{H\times H}$, by Theorem 4.39 we have that B also has vertex $\delta(D)$ so that B also has defect group D. Further, B and b are Green correspondents with respect to $H \times H$. As such, the map sending b to b^G is an injection from blocks of H with defect group D to blocks of G with the same defect group.

We need only show that all such blocks of G arise in this manner. Using Lemma 4.18 we see that $B_{H\times H}$ has some indecomposable summand with vertex $\delta(D)$. However, $B \mid kG$ and the only indecomposable summands of $(kG)_{H\times H}$ which have a vertex containing $\delta(D)$ are the summands of kH by Theorem 5.11 and these are the blocks of H. As such, there is a block b of H with defect group D such that $b \mid B_{H\times H}$ as required. ■

If $H = N_G(D)$ then we call the block b the *Brauer Correspondent* of b^G. This then yields b as the Green correspondent of the $k[G \times G]$-module b^G.

Theorem 5.17 (Brauer's Second Main Theorem)
Let V be an indecomposable kG-module lying in the block B of G. Let W be an indecomposable kH-module for $H \leq G$ lying in the block b of H with vertex Q such that $C_G(Q) \leq H$. If $W \mid V_H$ then b^G is defined and $b^G = B$.

We have a family of important corollaries to this theorem.

Corollary 5.18
If V is an indecomposable kG-module lying in the block B with vertex Q and corresponding indecomposable $kN_G(Q)$-module W lying in the block b then b^G is defined and $b^G = B$.

This approximately says that if V and W correspond via the Green correspondence then their blocks are Brauer correspondents.

Corollary 5.19
If B is a block of G with defect group D then there is an indecomposable kG-module lying in B with vertex D.

The proof of this corollary will not feature in lectures as it requires the Green correspondence.

Proof. Let b be the block of $N_G(D)$ corresponding to B and V be a simple $kN_G(D)$-module lying in b so that V is a $k[N_G(D)/D]$-module. Let P be a projective indecomposable module for $N_G(D)/D$

corresponding to V, so P also lies in b. Now, P is a summand of $k[N_G(D)/D] \cong \mathrm{Ind}_D^{N_G(D)} k$ so P is relatively D-projective and $P_D = (P_D)^D$. But then by Lemma 4.18 P_D has a summand with the same vertex as P, so P must have vertex D as the trivial module always has a Sylow p-subgroup as a vertex. Let U be the indecomposable kG-module corresponding to V via the Green correspondence. Hence, U has vertex D and lies in B by the previous corollary. ■

Corollary 5.20
The block B of G is a simple algebra if and only if B has defect zero.

We wish to include all three of Brauer's main three theorems in these notes, so we now give the third.

Theorem 5.21 (Brauer's Third Main Theorem)
If b is a block of $H \leq G$ with defect group D and $C_G(D) \leq H$ then $b^G = b_0(G)$ if and only if $b = b_0(H)$.

The proof of this theorem is given at the start of [1, §16] but we do not include it here as it makes use of tools that we haven't the time to introduce during this course.

6 Cyclic defect groups

We now take all of the tools we have been developing throughout the course and apply them to the case where the block B of G has a cyclic defect group. This situation occurs whenever the Sylow p-subgroups of G are cyclic. We have previously looked at the situation where G has a cyclic normal Sylow p-subgroup (Corollary 4.30) and $\mathrm{SL}_2(p)$ (Examples 3.38 and 4.31). However, it is actually possible to describe the structure of the projective modules for any block whose defect group is cyclic through the use of a structure known as a *Brauer tree*. The existence of Brauer trees is truly quite remarkable, as we shall see. We shall largely be following Alperin [1, Chapter V] for the start of this section.

Our first question, then, is clear: what is a Brauer tree? Unsurprisingly, it is a tree: a finite, connected, undirected graph containing no cycles or loops, but we also require a circular ordering of the edges incident at a given vertex and a distinguished vertex, called the *exceptional vertex* with a positive integer (called the *multiplicity* or occasionally *exceptionality*) associated to it. By the aforementioned circular ordering we mean that given a vertex v and an edge E incident at v, there is an edge which *comes after* E. One typically encodes this information by drawing the Brauer tree such that these circular orderings are obtained by simply travelling anti-clockwise about the vertex. Traditionally the exceptional vertex is drawn filled-in and all other vertices remain blank. We will draw all vertices the same way, and simply attach the multiplicity to the exceptional vertex on the graph itself. There are two particularly common instances of Brauer trees: *stars* and *lines*.

Definition 6.1
A *star* is a tree with n vertices, one of which has degree $n-1$ and all others have degree 1 (alternatively, one may regard this as a complete bipartite graph $K_{1,n-1}$). We may represent this as on the left in Fig. 1 where we have chosen the central vertex to be exceptional with exceptionality m.

Definition 6.2
A *line* (also called an *open polygon*) is a tree with two vertices of degree one and a Hamiltonian path between them (that is, a path including all vertices). We may represent this as on the right in Fig. 1 where there is no exceptional vertex (this is equivalent to the exceptional vertex having multiplicity 1, as we shall see).

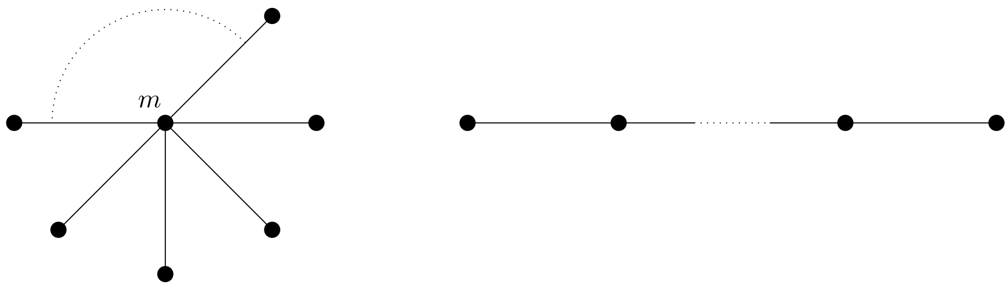


Figure 1: A star and a line.

Now, how exactly does the Brauer tree relate to the structure of our blocks? We do this via the below definition.

Definition 6.3
An algebra A is called a *Brauer tree algebra* if there is a Brauer tree so that the projective indecomposable A-modules are described by the graph as follows:

- There is a one-to-one correspondence between isomorphism classes of simple A-modules and the edges of the tree.
- If S is a simple A-module then $S \cong \mathrm{soc}(\mathcal{P}(S)) \leq \mathrm{rad}(\mathcal{P}(S)) \leq \mathcal{P}(S)$ and $\mathcal{H}(\mathcal{P}(S))$ is the direct sum of two (possibly zero) uniserial modules described by the orderings of the edges incident at the vertices at the endpoints of the edge corresponding to S as follows: Suppose that the edge corresponding to S has endpoints u and v with the edges incident at u (in order) correspond to the simple modules S, U_1, U_2, $\dots$, U_r and similarly the edges incident at v (in order) correspond to the simple modules S, V_1, V_2, $\dots$, V_n. If neither u nor v are exceptional, $\mathcal{H}(\mathcal{P}(S)) \cong U \oplus V$ where U and V are uniserial and have respective radical series U_1, U_2, $\dots$, U_r and V_1, V_2, $\dots$, V_n. If one of u or v is exceptional, without loss of generality we may assume that it is u with exceptionality m. Then the module U instead has radical series $U_1, U_2, \dots, U_r, S, U_1, U_2, \dots, U_r, S, \dots, U_{r-1}, U_r$ where each U_i appears as a composition factor of U with multiplicity m and S appears $m-1$ times.

The description above yields a picture of $\mathcal{P}(S)$ in the non-exceptional case as

$$\mathcal{P}(S) \sim \begin{matrix} S \\ U \oplus V \\ S \end{matrix} \qquad \text{where} \qquad U \sim \begin{matrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_r \end{matrix}, \quad V \sim \begin{matrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{matrix}$$

and when u is exceptional we instead have

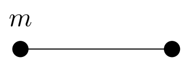
$$\mathcal{P}(S) \sim \begin{array}{c} S \\ U \oplus V \\ S \end{array} \qquad \text{where} \qquad U \sim \begin{array}{c} U_1 \\ U_2 \\ \vdots \\ U_r \\ S \\ U_1 \\ \vdots \\ U_r \\ S \\ U_1 \\ \vdots U_r \end{array}, \quad V \sim \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{array}$$

with S occurring as a composition factor of $\mathcal{H}(\mathcal{P}(S))$ $m-1$ times.

An easy way to remember how to determine the structure of the module U is as follows: Take a walk around the vertex u m times (where if u is not exceptional we set $m = 1$), starting just after the edge corresponding to S and ending just before returning to it.

We now look at some noteworthy special cases of Brauer trees.

Example 6.4
Suppose that A is a Brauer tree algebra for the below Brauer tree.

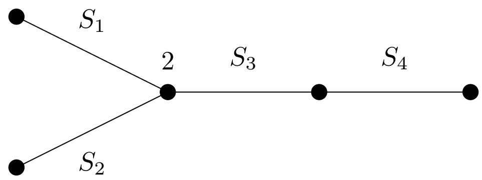


Then A has one simple module, S, and the uniserial module corresponding to the right vertex is zero, with the other module being uniserial of composition length $m-1$ and all composition factors isomorphic to S. Thus $\mathcal{P}(S)$ is uniserial with $m+1$ composition factors all isomorphic to S. If G is a cyclic group of order p^n then kG is a Brauer tree algebra for this Brauer tree with $m = p^n - 1$.

As we will now quite often need to talk about uniserial modules, we briefly introduce some notation to simplify matters and allow us to take up less space when doing so.

Definition 6.5
Suppose that V is a uniserial module with composition factors V_1, V_2, $\ldots$, V_n. Then we may describe the structure of V with the notation $V \sim [V_1 \mid V_2 \mid V_3 \mid \ldots \mid V_n]$.

Example 6.6
Suppose that A is a Brauer tree algebra for the below Brauer tree.

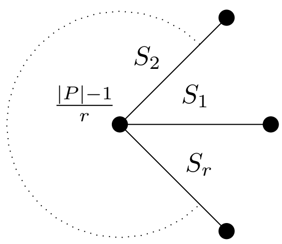


Then the projective indecomposable modules for such an algebra are as follows. We have $\mathcal{P}(S_1) \sim [S_1 \mid S_2 \mid S_3 \mid S_1 \mid S_2 \mid S_3 \mid S_1]$, $\mathcal{P}(S_2) \sim [S_2 \mid S_3 \mid S_1 \mid S_2 \mid S_3 \mid S_1 \mid S_2]$ and $\mathcal{P}(S_4) \sim [S_4 \mid S_3 \mid S_4]$ with the only non-uniserial projective indecomposable module being illustrated below with the heart of $\mathcal{P}(S_3)$ isomorphic to $U \oplus S_4$ where $U \sim [S_1 \mid S_2 \mid S_3 \mid S_1 \mid S_2]$..

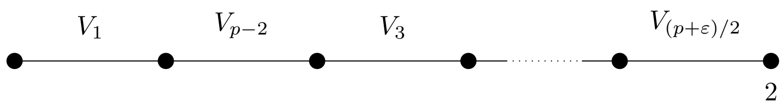
$$\mathcal{P}(S_3) \sim \begin{array}{ccc} & S_3 & \\ S_1 & & \\ S_2 & & \\ S_3 & \bigoplus & S_4 \\ S_1 & & \\ S_2 & & \\ & S_3 & \end{array}$$

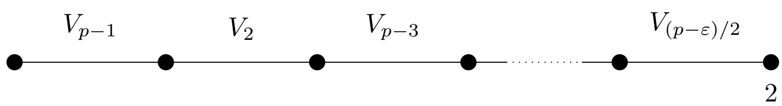
Let $G := \mathrm{Sz}(q) = {}^2\mathrm{B}_2(q)$ for $q = 2^{2n+1}$ and $n > 0$. Then $|G| = q^2(q-1)(q-s+1)(q+s+1)$ where $s^2 = 2q$ and G is a finite simple group ($\mathrm{Sz}(2)$ is still defined, but only has order 20 and is not simple). If $p \mid q-s+1$ then the principal block of kG is a Brauer tree algebra for the above Brauer tree with exceptionality (instead of the 2 used above) $(p^x-1)/4$ where p^x is the p-part of $q-s+1$.

Example 6.7
Now, recall from Corollary 4.30 and the results required to prove it that if G has a cyclic normal Sylow p-subgroup P, there is a 1-dimensional module W which we can use to determine the structure of the PIMs. One may also check that if B is a block of kG then we may choose notation so that $\mathrm{Irr}_k\, B = \{S_1, \ldots, S_r\}$ (where $\mathrm{Irr}_k\, B$ denotes the irreducible kG-modules lying in B) such that $S_{i+1} \cong S_i \otimes W$ with $S_r \otimes W \cong S_1$. This tells us that in fact B is a Brauer tree algebra for a star which has r edges and central exceptional vertex with exceptionality $(|P|-1)/r$. In particular, this Brauer tree may be drawn as below.



Example 6.8
Return to the case $G = \mathrm{SL}_2(p)$. As mentioned previously, for $p > 2$ there are three blocks for kG. In particular, the non-projective kG-modules $V_1, \ldots, V_{p-1}$ fall into two blocks which can be seen to be Brauer tree algebras with the following trees (with $p \equiv \varepsilon \mod 4$ for $\varepsilon = \pm 1$).





We now fix some notation. Let B be a block of G with cyclic defect group D of order p^n $(n \geq 1)$. Let b be the block of $N_G(D)$ which is the Brauer correspondent of B. The index of $DC_G(D)$ in the stabiliser in $N_G(D)$ of a block of $DC_G(D)$ which is covered by b is called the *inertial index* of B, denoted e. This is a well-defined number since all blocks of $DC_G(D)$ covered by b are $N_G(D)$-conjugate. The fundamental result for Brauer trees is the following.

Theorem 6.9
The block B is a Brauer tree algebra for a tree with e edges and exceptionality $(p^n - 1)/e$.

The proof of this result takes pretty much an entire chapter in [1], so of course we do not have room for it here. We can, however, provide a brief overview of a route to the proof. Let D_1 be the subgroup of D of order p and let $N_1 := N_G(D_1)$. Let $b = b_1^{N_1}$ so that b_1 is a block of N_1 with defect group D and $b_1^G = B$. The first step to the proof of the above theorem is to study b_1 and in particular prove the following.

Theorem 6.10
The block b_1 is a Brauer tree algebra for a star with e edges and central exceptional vertex with exceptionality $(p^n - 1)/e$.

Once the structure of b_1 itself is known, it is then necessary to use this to obtain information about the structure of B via the following theorem which (with some work) follows from the Green correspondence.

Theorem 6.11
There is a one-to-one correspondence between isomorphism classes of non-projective indecomposable B-modules and isomorphism classes of non-projective indecomposable b_1-modules such that if U and V are corresponding such B and b_1-modules, respectively, then

$$\begin{aligned} \operatorname{Ind}_{D_1}^G V &\cong U \oplus Q \\ U_{N_1} &\cong V \oplus W \end{aligned}$$

where Q is a projective kG-module and W is a direct sum of a projective kN_1-module and modules which lie in blocks of N_1 other than b_1.

Using the above theorem the proof then roughly proceeds as follows: Prove that $|\operatorname{Irr}_k B| = e$, where e is the inertial index of B, then looking at the possible extensions between these irreducible modules show that B is a Brauer *graph* algebra (where a Brauer graph is broadly the same as a Brauer tree, except not necessarily a tree and may have more than one exceptional vertex), and from this one then concludes that the Brauer graph in question must be a Brauer tree.

Obviously, we have skipped over a lot of detail — in [1] the proof of the theorem spans about 50 pages! We shall instead investigate some related results.

Since any block with a cyclic defect group has a Brauer tree, a natural question to ask is: which trees can be the underlying graph for a Brauer tree? This question was answered by Feit [10] in the 80s.

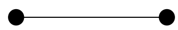
Definition 6.12
Let τ be a tree and let v_0 be a vertex of τ and $n \in \mathbb{N}$. Then we define $(\tau, v_0)^n$ to be the union of n copies of τ with the vertices v_0 identified. Now, two trees τ and σ are said to be *similar* if there exists a tree γ such that $\tau \cong (\gamma, v_0)^n$ and $\sigma \cong (\gamma, v_0')^m$ for some vertices v_0, v_0' of γ and positive integers m, n.

One way to regard this definition is to say that two trees are similar if they can both be obtained by sticking together many copies of some other tree.

The first main result of Feit's paper is as follows, where we say that a tree τ *belongs* to a group G if there exists a block B of G such that τ is the underlying graph of a Brauer tree for B.

Theorem 6.13 [10, Theorem 1.1]
Let G be a finite group and let τ be a tree belonging to G. Then there exists a simple group H involved in G and a group $\tilde{H}$ where $H = \tilde{H}$ if $|H| = p$ and otherwise $\tilde{H}$ is perfect with $\tilde{H}/Z(\tilde{H}) \cong H$ such that τ is similar to a tree that belongs to $\tilde{H}$. Further, if $\tau = (\gamma, v_0)^n$ for some $n > 1$ then v_0 is the exceptional vertex of τ if it has one.

Broadly speaking, this tells us that to determine the trees which may appear as Brauer trees for blocks of finite groups, it is sufficient to determine the Brauer trees of blocks of finite simple and quasisimple groups. Also, in the case where G is p-soluble (all nonabelian composition factors of G have order coprime to p) this tells us that τ must be similar to the tree



and so in particular must be a star with exceptional vertex in the centre.

Making use of the classification of finite simple groups, it was then possible to prove the following.

Theorem 6.14 [10, Theorem 1.2]
Let G be a finite group and let τ be a tree belonging to G. Then $\tau \cong (\gamma, v_0)^n$ for some $n \in \mathbb{N}$ where v_0 is the exceptional vertex (if there is one). Further, either γ has at most 248 edges or γ is a line.

So we see that any Brauer tree of a block of a finite group is either similar to a 'small' graph or to a line.

We now take a brief diversion to explore what, specifically, makes the case of blocks with cyclic defect groups so different to the general case. For that, though, it is helpful to introduce some definitions.

At this point, we reiterate our assumption that the field k is algebraically closed (though there are still things to be done at least for group algebras over non–algebraically closed fields, they complicate matters).

Given an algebra A, there is a notion of *representation type* for A which, broadly speaking, describes how complicated the representation theory of this algebra is.

Definition 6.15
Let A be an algebra. Then A is said to be of *finite representation type* if there are only finitely many isomorphism classes of indecomposable A-modules. Otherwise, the representation type of A is *infinite*.

Among those algebras of infinite representation type, we further distinguish between those that are in some sense manageable, and those that are not. The formal definitions of these properties are quite complicated! For the next definition, note that an A-B-*bimodule* is simply a left A-module M which is also a right B-module such that the actions of A and B are compatible, so $(am)b = a(mb)$ for $a \in A$, $b \in B$ and $m \in M$. Also recall that $k[X]$ denotes the polynomial algebra in X over k.

Definition 6.16
Let A be an algebra. Then the representation type of A is *tame* if it is not finite and for any integer $d > 0$ there are a finite number of $k[X]$-A-bimodules M_i which are free as left $k[X]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to $N \otimes_{k[X]} M_i$ for some i and some simple $k[X]$-module N.

Broadly speaking, this simply says that there exists a 'reasonable' parameterisation of the indecomposable A-modules. These tame algebras are those algebras of infinite representation type which are in some sense manageable, which leaves us with those that are not. Unsurprisingly, this definition is also complicated and requires the use of terms that we haven't the room to formally define in this course. Here $k\langle X, Y\rangle$ is the free algebra in two generators over k, which is akin to the polynomial algebra except that we do not require that the indeterminates X and Y commute.

Definition 6.17
Let A be an algebra. Then the representation type of A is *wild* if there is a finitely generated $k\langle X, Y\rangle$-A-bimodule B which is free as a left $k\langle X, Y\rangle$-module such that the functor $- \otimes_{k\langle X,Y\rangle} B$ from mod-$k\langle X, Y\rangle$ to mod-A preserves indecomposability and reflects isomorphisms.

Roughly, (and, vitally, attempting to avoid any actual discussion of category theory) the above states that the representation theory of the free algebra $k\langle X, Y\rangle$ may be embedded into the representation theory of any wild algebra. It follows from this, in fact, that the representation theory of any algebra B may then be embedded into the representation theory of any wild algebra A. (At this point it is helpful to recall that all of our algebras are finite-dimensional)

Now, we presented these representation types as three distinct options but it is obviously necessary to prove as much. The below theorem was originally proven by Drozd (Russian: [9], 1986 English translation: [15]) but there is also a 1987 proof in English due to Crawley-Boevey [4].

Theorem 6.18 (Trichotomy theorem)
Let A be an algebra over an algebraically closed field k. Then the representation type of A is either finite, tame or wild.

Now, the reason we introduce these results is, of course, because we would like to apply them to the case of group algebras.

Theorem 6.19
Let G be a finite group and B a block of kG. Suppose that $D \in \mathrm{Syl}_p(G)$ or, respectively, D is a defect group of B. Then the representation type of kG, respectively of B, is

i) finite if D is cyclic.

ii) Tame if $p = 2$ and D is dihedral, semidihedral or generalised quaternion.

iii) Wild, otherwise.

The fact that the representation type of kG (or a block) is tame when the Sylow p-subgroups (or defect groups) are so specific is largely due to a result of Brenner [2] which shows that the representation type of a non-cyclic p-group P is wild unless $[P\colon P'] \leq 4$. However, for a p-group P of order at least p^2 we have that the index of the commutator subgroup of P is always at least p^2, so the only possibilities for tame p-groups are 2-groups. It in fact turns out that the 2-groups for which the representation type is tame are precisely the 2-groups of *maximal class*, which are the three types of 2-group listed above.

Now, the purpose for this diversion is the following observation: If a block B has a cyclic defect group then there are finitely many isomorphism classes of indecomposable modules in B. Since there are only finitely many it thus makes sense to ask: can we determine what they are? It turns out, the answer is yes. The following exposition largely follows a paper of Janusz [13, §5].

The below is true for any Brauer tree algebra, but we shall state results only for group algebras for simplicity.

Theorem 6.20
Let Γ be a Brauer tree with exceptional vertex of exceptionality $m \geq 1$ and let B be a block with Brauer tree Γ. For an edge E of Γ let V_E denote the irreducible kB-module corresponding to the edge E.

i) Let E, F be edges in Γ both incident at a common vertex v. If v is not exceptional then let $n = 1$, otherwise let $1 \leq n \leq m$. Then there is a unique uniserial module $M = M(F, E; n)$ such that $V_F \cong \operatorname{head} M$ and $V_E \cong \operatorname{soc} M$ with V_E and V_F both having multiplicity n as a composition factor of M.

ii) Now let E be an edge containing the exceptional vertex and let $2 \leq n \leq m$. Then there is a unique uniserial module $M = M(E, E; n)$ such that $\operatorname{head} M \cong \operatorname{soc} M \cong V_E$ and V_E appears as a composition factor of M with multiplicity n.

Further, each of the modules listed in i) and ii) are reducible and non-projective, and any reducible, non-projective uniserial kB-module is isomorphic to exactly one such module.

Picking two vertices v_0 and v_{k+1} in Γ, we are interested in the path between these two vertices. These paths may take one of two forms. The most obvious form is below, where we take a direct path between the two vertices as in Fig. 2.

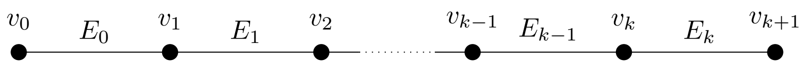


Figure 2: A direct path.

If the exceptional vertex v_e is present then it may be used as a turning point and so the path may have a branch in it. This branch point, denoted v_h, can be any non-exceptional vertex in the path illustrated in Fig. 3. In particular, it is possible for $h = -1$ or $h + 2t = k$, or both!

Now, given a chain of edges $E_1, \ldots, E_k$ lying in one of the two cases illustrated in Figs. 2 and 3, we define the set D to be either all of the even integers in $\{0, 1, \ldots, k\}$ or all of the odd integers

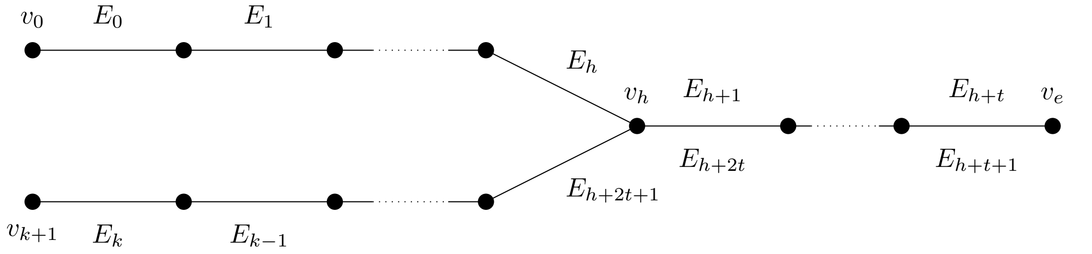


Figure 3: A branching path containing the exceptional vertex v_e.

in this set (note as $k \geq 1$ at least $D \neq \varnothing$). For each $1 \leq i \leq k$, choose a uniserial module M_i such that $M_i \cong M(E_i, E_{i-1}; n_i)$ for $i \in D$ and $M_i \cong M(E_{i-1}, E_i; n_i)$ if $i \notin D$ for $M(E, F; n)$ as described in Theorem 6.20.

If we are in the case of Fig. 2 and none of the vertices are exceptional then $n_i = 1$ for all i, otherwise $1 \leq n_j \leq m$ if the vertex v_j is exceptional. In the other case, as in Fig. 3, then $n_i = 1$ provided $i \neq h + t + 1$ and $2 \leq n_{h+t+1} \leq m$. In particular, there is at most one i such that $n_i \neq 1$.

For each i, fix $V_i \cong \operatorname{head} M_i$ and $W_i \cong \operatorname{soc} M_i$ such that if $V_i \cong W_i$ then $V_i = W_i$. Now, for each i we may choose a surjective homomorphism $\varphi_i \colon M_i \to V_i$ and an injective homomorphism $\psi_i \colon W_i \to M_i$ and make the following definitions.

$$X \coloneqq \{(m_1, m_2, \ldots, m_k) \in \bigoplus_{i=1}^{k} M_i \mid \varphi_i(m_i) = \varphi_{i+1}(m_{i+1}) \text{ for all } i \in D,\ 0 < i < k\},$$

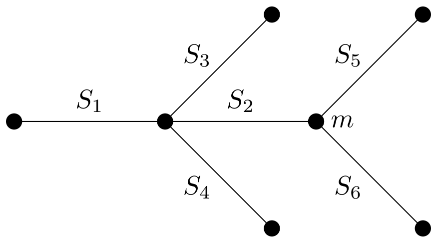
$$Y \coloneqq \left\{ \sum_{\substack{i \in D \\ i > 1}} \psi_{i-1}(f) \oplus \psi_i(f) \mid f \in V_{E_{i-1}} \right\} \leq \bigoplus_{i=1}^{k} M_i,$$

finally, we define $W \coloneqq X/Y$.

Note that in the definition of Y we mean that $\psi_i(f)$ should be regarded as the element of $\bigoplus_{i=1}^{k} M_i$ with all entries zero except in the ith position.

At first glance, it is not entirely clear what such a module W really looks like. We shall illustrate this with an example.

Example 6.21
Let Γ be the below Brauer tree. (This is the Brauer tree for the principal block of a Suzuki group $\mathrm{Sz}(q)$ when p is an odd prime dividing $q + 1$)



Take the path in Γ consisting of the edges labelled by S_1, S_2 and S_5, in that order. Then this is a path as in Fig. 2 with $k = 2$ and the edge E_0 corresponding to S_1, E_1 corresponding to S_2 and E_2 corresponding to S_5. Set $D \coloneqq \{0, 2\}$. Then we have $M_1 \coloneqq M(S_1, S_2; 1)$ and $M_2 \coloneqq M(S_5, S_2; n)$ for some $1 \le n \le m$. We also have that M_1 and M_2 are both uniserial with $M_1 \sim [S_1 \mid S_2]$ and $M_2 \sim [S_5 \mid S_2 \mid S_6 \mid S_5 \mid S_2 \mid \cdots \mid S_2]$ where the section $[S_6 \mid S_5 \mid S_2]$ of M_2 appears $n - 1$ times.

Now, since $D = \{0, 2\} = \{0, k\}$ we actually have that the condition on X is empty and thus $X = M_1 \oplus M_2$. Further, the module Y is the diagonal submodule $\{\psi_1(f) \oplus \psi_2(f) \mid f \in S_2\} \le S_2 \oplus S_2$ which we may regard as the submodule $\{(s, s) \mid s \in S_2\} \le S_2 \oplus S_2$. Thus $Y \le \operatorname{soc}(M_1 \oplus M_2)$ is not contained in either M_1 or M_2. One can see that $\operatorname{soc}(X/Y) \cong (\operatorname{soc} X)/Y \cong S_2$ and so we have that $W = X/Y$ has socle S_2 such that $W/\operatorname{soc} W \cong M_1/\operatorname{soc} M_1 \oplus M_2/\operatorname{soc} M_2 \cong S_1 \oplus [S_5 \mid S_2 \mid S_6 \mid S_5 \mid \cdots \mid S_5]$ and so we obtain the below picture of W.

$$W \sim \begin{array}{cc} & S_5 \\ & S_2 \\ & S_6 \\ S_1 \oplus & S_5 \\ & \vdots \\ & S_6 \\ & S_5 \\ S_2 & \end{array}$$

Note that since the socle of W is irreducible, we in fact have that W may be realised as a submodule of $\mathcal{P}(S_2)$.

Example 6.22
Let Γ be the Brauer tree from Example 6.21 and take instead the path in Γ consisting of the edges, in order, labelled by S_1, S_2, S_2, S_3. Then this is a path as in Fig. 3 with $k = 3$ and the edge E_0 corresponding to S_1, the edges E_1 and E_2 corresponding to S_2 and the edge E_3 corresponding to S_3. Set $D \coloneqq \{1, 3\}$. As before, we have $M_1 \coloneqq M(E_1, E_2; 1)$, $M_2 \coloneqq M(E_1, E_2; n)$ for some $2 \le n \le m$ and $M_3 \coloneqq M(E_3, E_2; 1)$ so that $M_1 \sim [S_2 \mid S_4 \mid S_1]$, $M_3 \sim [S_3 \mid S_1 \mid S_4 \mid S_2]$ and $M_2 \sim [S_2 \mid S_5 \mid S_6 \mid S_2 \mid \ldots \mid S_6 \mid S_2]$ with the section $[S_5 \mid S_6 \mid S_2]$ appearing $n - 1$ times.

In this case, then, we have

$$X \coloneqq \{(m_1, m_2, m_3) \in M_1 \oplus M_2 \oplus M_3 \mid \varphi_1(m_1) = \varphi_2(m_2)\}.$$

Clearly we have that $\operatorname{rad} M_1 \oplus \operatorname{rad} M_2 \oplus M_3 \le X$ and so, up to twisting by some automorphism of S_2, we have that X is the preimage under the quotient map of some diagonal submodule

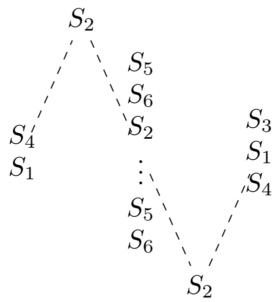
$\{(x,x) \mid x \in S_2\} \leq S_2 \oplus S_2$, yielding a picture of X/M_3 as follows.

$$X/M_3 \sim \begin{array}{ccc} & S_2 & \\ & & S_5 \\ & & S_6 \\ S_4 & & S_2 \\ S_1 & \oplus & \vdots \\ & & S_6 \\ & & S_2 \end{array}$$

and similarly to before we see that X/M_3 has a simple head and is thus a quotient of $\mathcal{P}(S_2)$.

$$Y = \{\psi_2(g) \oplus \psi_3(g) \mid g \in S_2\}$$

and again we have that Y is an irreducible diagonal submodule of $\mathrm{soc}(M_2 \oplus M_3) \cong S_2 \oplus S_2$. Collecting this information, we see that $W := X/Y$ is as seen below, where the dashed lines indicate a non-split extension between the indicated indecomposable modules.



From the above, it is possible to get an idea for what happens in the general case: One takes uniserial modules with isomorphic heads and takes the preimage of some diagonal submodule of the head of the direct sum to 'stick together' the modules. One may also do the same through taking diagonal submodules of the socle of such modules, and alternating these processes one can make long chains of uniserial modules which are 'attached' by simple modules in their socles and heads.

Indeed, the below theorem shows that all indecomposable modules for blocks with cyclic defect groups arise in this manner.

Theorem 6.23
For any module W defined as above, we have the following.

i) The isomorphism type of W is independent of the choice of the isomorphisms φ_i, ψ_i.

ii) Each such module W is indecomposable.

iii) Every non-projective indecomposable A-module is either irreducible or isomorphic to a module W as constructed above.

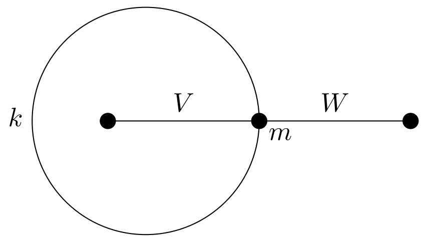
We now finish with the briefest look at what may happen in the non-cyclic case. We have already seen in Example 3.25 that for $G = C_p \times C_p$ there are infinitely many indecomposable kG-modules, and in fact by Theorem 6.19 we may infer that it is likely much more complicated than just this. We shall instead look at a case that may not be quite so bad.

Example 6.24
Let $G \coloneqq \mathrm{PSL}_2(q)$ for $q \equiv 1 \mod 4$ a prime power and suppose that $p = 2$. Let $P \in \mathrm{Syl}_p G$. Then P is dihedral of order $(q-1)_2$ (recalling that n_p denotes the p-part of n) and so by Theorem 6.19 we have that the representation type of G is tame. In fact, by work of Donovan and Freislich [8] we know that the projective indecomposable kG-modules may be described by a *Brauer graph* (not a tree!). Let B_0 be the principal block of kG and, as usual, let k denote the trivial kG-module. Let $q = r^a$, $R \in \mathrm{Syl}_r(G)$ and let $B \coloneqq N_G(R)$. Then B (often called a *Borel subgroup*) may be regarded as the image in G of the upper triangular matrices from $\mathrm{SL}_2(q)$. Then the permutation module of G acting on cosets of B has shape

$$\mathrm{Ind}_B^G k \sim \begin{array}{c} k \\ V \oplus W \\ k \end{array}$$

for irreducible kG-modules V, W each of dimension $\frac{1}{2}(q-1)$. It turns out that the only irreducible modules in the principal block of kG are k, V and W.

The Brauer graph of the principal block in this case is given below. Brauer graphs function exactly the same way as Brauer trees, though they clearly needn't be trees and there needn't be a unique exceptional vertex with multiplicity greater than 1.



Of course, in this case there is still only one vertex with multiplicity greater than one, but the above graph is definitely not a tree. We may, however, still go on as usual and work out the structure of the PIMs in this case. For clarity, once a complete cycle around a vertex has been completed, we have indicated in bold when a new walk around that vertex has begun. Since the vertex with multiplicity m is connected to all edges of the graph, every PIM has multiple repeated composition factors (and we can see that in fact k appears in every second radical layer of every PIM in the principal block here!).

$$
\begin{array}{c}
V \\ k \\ W \\ k \\ \boldsymbol{V} \\ \vdots \\ W \\ k \\ V
\end{array}
\qquad\qquad
\begin{array}{c}
W \\ k \\ V \\ k \\ \boldsymbol{W} \\ \vdots \\ V \\ k \\ W
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & k & \\
V & & W \\
k & & k \\
W & & V \\
\boldsymbol{k} & \bigoplus & \boldsymbol{k} \\
\vdots & & \vdots \\
k & & k \\
W & & V \\
 & k &
\end{array}
$$

In general, to construct indecomposable modules for such a group one may proceed similarly to the above treatment given by Janusz and stick together modules as one walks along the Brauer graph, but since the graph needn't be a tree and there can be multiple vertices with multiplicity greater than one, it is possible to create infinite cycles as one travels through the graph and thus generate infinitely many indecomposable modules along the way.

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